

EURASIAN JOURNAL OF BUSINESS AND MANAGEMENT

www.eurasianpublications.com

LIQUIDITY ADJUSTED VALUE AT RISK: INTEGRATING THE UNCERTAINTY IN DEPTH AND TIGHTNESS

Levent C. Uslu

Corresponding author: Yeditepe University, Turkey. Email: cluslu@yeditepe.edu.tr

Burak Evren

Riskturk Risk Software Technologies, Turkey. Email: burak.evren@riskturk.com

Abstract

Efficient market risk management should also focus on the market liquidity risk, which is generally ignored by the conventional Value at Risk Metrics. We propose two alternative parametric methods to existing studies for the estimation of Liquidity Adjusted Value at Risk (LVaR). The first model is based on the volatility dynamics of VNET (Engle and Lange, 2001) whereas the second model also incorporates the first two moments of the tightness dimension to the latter, as measured by relative weighted bid-ask spreads. Considering a portfolio with different underlying volatility assumptions (EWMA, GARCH-CCC, GARCH-DCC), validation results indicate that both parametric LVaR approaches are strong alternatives to tightness based LVaR models and strictly superior to conventional VaR models, with respect to performance related to regulatory compliance, statistical coverage and overall relative cost of liquidity vs. loss size on violation days.

Keywords: Liquidity Adjusted Value at Risk, Cost of Liquidity, VNET, Depth Uncertainty

JEL classification: G10, G12, G17, G32

1. Introduction

Market risk measurement involves modeling the changes in asset values throughout the time which constitutes the corner stone of risk management. Initiated by the Basel Regulations, Value at Risk (VaR) is recently the most widely used technique to measure market risk in finance industry. However in practice, the main drawback of conventional VaR methods is that they ignore the important dimensions of market liquidity such as the level of depth, tightness and resiliency. Depth refers to the maximum number of tradable assets without having a significant impact on the market price. Tightness or width is a general term used for the transaction costs incurred by the distance (spread) between bid and ask quotes.¹ Lastly,

¹ Although Bid-ask spreads may reflect both information asymmetry and inventory costs, we restrict ourselves with the prior one, hence assume market without any market maker, as our sample data is taken from order driven electronic market.

resiliency is the market's ability or speed in correcting temporary mispricing and order imbalances. Clearly, conventional VaR methods ignore liquidity dynamics by assuming infinite depth and relying on the liquidation with no additional cost (e.g. cost of liquidity). Thus, these models are highly exposed to understated risk forecasts, especially in times of financial turmoil. An efficient market risk measure should certainly involve liquidity information, which is also emphasized by the Basel III regulations and motivates the recent research on diverse liquidity-adjusted VaR (LVaR) models.²

Past and recent research for the integration of liquidity risk into market risk measurement fall under two main model classes, namely endogenous or exogenous models. Endogenous models agree on the realistic assumption that size of an asset position matters for the market liquidity; that is a small investor trying to sell a couple of shares in the stock market would have less impact on the current market price as compared to an institutional investor placing a large block of sell order. Although, endogenous models are highly sophisticated and have broader perspective than that of exogenous counterparts and they generally focus on optimal execution strategies with fixed or floating intraday time horizons to solve for the liquidation problem. More importantly, even though the computer technology is recently capable of handling large datasets, majority of these models are less straightforward to be implemented by finance practitioners, either due to time or hardware limitations or complexity or to some extent due to data availability/quality. On the other hand, exogenous models are straightforward in implementation and interpretation but come with a simplifying and often criticized assumption, implying a perfect competition in market, whereby no single investor would have significant impact on the market price (Bangia *et al.* 1999; Stange and Kaserer, 2008; Ernst *et al.* 2012).

In this study we strict our attention to conventional inter-day market risk measurement (VAR) by using intraday order book and transaction data and contribute to the existing literature for the LVaR models with two alternative approaches. The first model relies on the estimation of the volatility of daily close-to-close log returns by using the volatility of uncertainty involved in the market depth, from which we expect to capture the level of order imbalance and resiliency dynamics in the market. In order to account for the uncertainty in the market depth, we use the net trading volume (VNET) indicator as an input, which was proposed by Engle and Lange (2001). The second model integrates also the tightness dimension of liquidity measured by the first two moments of weighted relative bid-ask spread as the proxy variable, similar to previous studies (Le Saout, 2002; Francois-Heude and van Wynendaele, 2001; Giot and Grammig, 2002; Stange and Kaserer, 2008; Qi and Wing, 2009). Results show that both parametric LVaR approaches are strictly superior to conventional Value at Risk (VaR) models, with respect to performance in terms of regulatory compliance and statistical coverage and the relative cost of liquidity vs. loss size given violation.

The remainder of the paper is organized as follows: Section 2 describes the data and methodology. Section 3 is reserved for the review of previous studies focusing on LVaR models. Finally, Section 4 shows our main results and Section 5 concludes.

2. Data and Methodology

2.1. Data

We construct an equally weighted portfolio of 10 banking index equities (Table 1) and for each equity we use intraday order book and transaction data obtained from Borsa Istanbul (BIST), from where we extract daily weighted bid-ask prices and VNET volatilities.³ The data was filtered and cleared from the automatically cancelled orders for each trading day t , prior to our

² In particular Basel Framework also encourages the use of intraday high-frequency data for early warning regarding market illiquidity.

³ The reason we choose banking index equities needs to be justified: When compared to equities from other industry indexes, banking equities generally exhibit relatively higher trading volume and expected to respond to market news more rapidly due to fragile nature of banking balance sheets. In this manner, we aim to overcome the market thinness that is often observable in emerging stock markets and obtain more information from data as possible.

analyses. Moreover, our VaR/LVaR estimation sample is between 15/10/2009 and 05/30/2014 (1158 trading days), whereas the back testing sample involves 908 observations.

Table 1. Equities (Portfolio)

Equity (Ticker)	Definition
GARAN	Garanti Bankasi
AKBNK	Akbank
ISCTR	Turkiye Is Bankasi
VAKBN	Vakifbank
HALKB	Turkiye Halk Bankasi.
YKBNK	Yapi ve Kredi Bankasi
FINBN	Finansbank
ASYAB	Asya Katilim Bankasi.
SKBNK	Sekerbank
TEKST	Tekstilbank

Source: Borsa Istanbul

2.2. Measuring the Uncertainty in Market Depth

Engle and Lange (2001) proposed $VNET$ as a measure of realized depth for a specific price deterioration, which is based on excess volume of buys/sells that trigger a price duration.⁴ If N_h is the total number of transactions within a price duration h , then the corresponding $VNET$ for any price duration can be calculated as:

$$VNET_{h,it} = \log \left| \sum_{j=1}^{N_h} d_j V_j^{Trade} \right| \quad (1)$$

Here, d_j is the trade direction and V_j^{Trade} is the number of assets traded at the j 'th h time point within a price duration h . In order to determine d_j , we use the Lee and Ready Algorithm (Lee and Ready, 1991) which is widely accepted both by previous and recent studies investigating the transaction book dynamics (O'Hara *et al.* 2000; Chakrabarty *et al.* 2012).⁵ Finally, after calculating the absolute net volume at each price duration point, the end-of-day volatility of $VNET$ or the level of market depth uncertainty (MDU) for any given asset l at day t with m price durations can be calculated as:

$$MDU_{i,t} = \frac{1}{m} \sqrt{\sum_{h=2}^m (VNET_{h,it} - VNET_{h-1,it})^2} \quad (2)$$

An important point needs to be justified here that the calculation of $VNET$ points does not depend on any specific time frequency and solely focuses on the irregular time intervals where price durations occur. As a consequence of this event-based measure, converting the duration points into equally spaced time intervals and using interpolation techniques might lead to biased $MDU_{i,t}$, especially when there are few intraday price variations, therefore we take the realized volatility as formulated above by using unequally spaced price duration points.

$MDU_{i,t}$ must be investigated in two dimensions. The first dimension is the price formation frequency: If we assume the difference between consecutive excess realized log depth ($VNET_{h,it} - VNET_{h-1,it}$) levels to be constant for all $h=1, \dots, m$, at any day t , then any increase in the intraday variation of market price (e.g. realization of limit orders at various price levels), would lead to a decrease in $MDU_{i,t}$. Therefore, being other things equal, an increase in the price movement will lead to decreasing uncertainty in the market depth as captured by $MDU_{i,t}$. The second dimension is the variation in the size of $VNET_{h,it}$. Obviously, if the number

⁴ Price duration can be defined as the length of time necessary to move the market price one tick regardless of the direction.

⁵ Please refer to Lee and Ready (1991) for details.

of intraday price levels is assumed to be constant for any day t , an increasing (decreasing) magnitude of $VNET_{h,it}$ (order imbalance) would lead to higher (lower) $MDU_{i,t}$. Thus, as it will be seen in the following subsections, by using the variation in $MDU_{i,t}$ as a predictor for daily return variance, we aim to incorporate the unexplained intraday depth-based liquidity dynamics into conventional VaR metric.⁶

2.3. Measuring the Market Tightness

As market tightness for any asset i , we refer to the transaction cost indicated by the size of the spread between bid and ask quotes. Similar to previous studies (Le Saout, 2002; Francois-Heude and van Wynendaele, 2001; Giot and Grammig, 2002; Stange and Kaserer, 2008; Qi and Wing, 2009) we use the weighted spread in order to incorporate the endogenous effects. On the other hand, our spread calculation methodology differs slightly from the existing studies: we take price duration points as intervals for spread calculation, without relying on any specific intraday time interval, which is in line with the calculation methodology of $VNET_{h,it}$ as described before. Therefore, we focus on price dimension instead of time dimension and for each price duration h we obtain the weighted ask ($A_{h,it}^w$) and bid prices ($B_{h,it}^w$) with corresponding order volumes ($V_{A,j}, V_{B,j}$) as follows:

$$A_{h,it}^w = \sum_{j=1}^{N_h} A_j / V_{A,j} \quad (3)$$

$$B_{h,it}^w = \sum_{j=1}^{N_h} B_j / V_{B,j} \quad (4)$$

Next, by considering m price durations we determine the weighted daily ask (A_{it}^w) and bid prices (B_{it}^w) as:

$$A_{it}^w = \sum_{h=1}^m A_{h,it}^w, \quad (5)$$

$$B_{it}^w = \sum_{h=1}^m B_{h,it}^w \quad (6)$$

The final step is to define the daily spread. As also suggested by Bangia *et al.* (1999), in order to eliminate the size effects incurred by different asset prices we use relative weighted spread (SPR_{it}) which is a normalization procedure by using the mid-price⁷:

$$SPR_{it} = \frac{(A_{it}^w - B_{it}^w)}{(A_{it}^w + B_{it}^w)/2} \quad (7)$$

2.4. VaR Framework

2.4.1. Parametric VaR

We can define VaR briefly as the potential loss of the value of a portfolio or an asset, given a predetermined time horizon and confidence level.⁸ By using matrix notation, for a portfolio of n assets, the conventional parametric VaR estimation for day $t+1$ can be formulated as

⁶ In this manner, $MDU_{i,t}$ can be seen as the average intra day level of uncertainty of depth for any asset i .

⁷ The average of A_{it}^w and B_{it}^w

⁸ VaR can be predicted by using parametric (eg. Variance Covariance, Monte Carlo), semi parametric (eg. Filtered Historical simulation) as well as non-parametric approaches (eg. Historical Simulation). In this study, we prefer to focus on the easy-to- implement parametric Variance-Covariance approach, given that the risk factors are equities with linear dynamics, which is not the case for complex derivative instruments.

$$VaR_{t,\alpha} = -\mu_{p,t} + z_\alpha \sqrt{[w_{1,t}, w_{2,t}, \dots, w_{n,t}] \Sigma_t \begin{bmatrix} w_{1,t} \\ w_{2,t} \\ w_{n,t} \end{bmatrix}} = -\mu_t + z_\alpha \sqrt{H_t}$$

where, (8)

$$\Sigma_t = \begin{bmatrix} \sigma_{11,t}^2 & \dots & \sigma_{1n,t} \\ \vdots & \sigma_{22,t}^2 & \vdots \\ \sigma_{n1,t} & \dots & \sigma_{nn,t}^2 \end{bmatrix}, \text{ and } H_t = w_t' \Sigma_t w_t$$

In Eq.8, μ_t is the portfolio's expected return, $w_{i,t}$ are asset weights and $\sigma_{jj,t}^2$ and $\sigma_{ij,t}$ ($i,j=1, \dots, n$) stand for variances and covariance between daily log asset returns, respectively. Lastly, H_t represents the estimated portfolio variance until day t , whereby α is the confidence level and z_α is the underlying inverse probability function.⁹

2.4.2. Volatility Estimation

Since our VaR method is parametric, the most crucial step is to determine the underlying volatility estimation process (H_{t+1}). For this purpose, we follow two widely used approaches, namely EWMA¹⁰ and GARCH (1,1)¹¹ processes. For a portfolio of assets we get the following EWMA estimation.¹²

$$H_{E,t+1} = \lambda H_{E,t} + (1 - \lambda)(R_t' R_t) \tag{9}$$

In Eq.9 GARCH framework is separable in terms of variance and correlation estimations. This opportunity comes with two popular methods. As proposed by Bollerslev (1990), the constant conditional correlation model (CCC) assumes that correlations are constant over time, whereas the dynamic conditional correlation model (DCC) as proposed by Engle (2002) as well as by Tse and Tsui (2002), assumes that correlations may evolve through time.¹³ If we denote the sample correlation matrix at time t as P_t and V_t as the diagonal matrix of time varying volatilities of portfolio assets:

$$V_t = \begin{bmatrix} \sigma_{1,t} & \dots & 0 \\ \vdots & \sigma_{i,t} & \vdots \\ 0 & \dots & \sigma_{n,t} \end{bmatrix} \tag{10}$$

Then we obtain the GARCH-CCC covariance estimate for day $t+1$ as follows:

⁹ Invalidation tests, we include both $\alpha=0.01$ and $\alpha=0.05$, corresponding to 0.99 and 0.95 confidence levels, respectively. Note also that, Variance-Covariance Approach is based on the assumption of standard normal distribution. Moreover, log returns are demeaned at each estimation in stance t , therefore μ_t can be set to 0. Readers should refer to Jorion (2006) and Alexander (2008) for details regarding the model assumptions behind VaR methods.

¹⁰ EWMA stands for the Exponentially Weighted Moving Average Process, introduced by Riskmetrics® (1996). Daily EWMA is estimated by $\lambda=0.94$. See Riskmetrics® (1996) for further details.

¹¹ Generalized Autoregressive Conditional Heteroscedasticity with 1 autoregressive and 1 moving average component. See Engle (1982) for details.

¹² GARCH estimation is similar to EWMA, except that EWMA ignores the mean reversion parameter a and also sets $b + c = 1$, by letting the smoothing constant λ to measure the decay speed.

$$H_{GC,t+1} = V_t P_t V_t' \quad (11)$$

GARCH-DCC generalizes the GARCH-CCC approach with the following properties:

$$H_{GD,t+1} = V_t \text{diag}(Q_t)^{-1/2} Q_t \text{diag}(Q_t)^{-1/2} V_t' \quad (12)$$

$$Q_t = \bar{Q}(1 - \zeta - \xi) + \zeta \varepsilon_{t-1} \varepsilon_{t-1}' + \xi Q_{t-1} \quad (13)$$

In Eq.13, Q_t is the positive definite autoregressive matrix and ε_{t-1} are the standardized residuals.¹⁴ $H_{E,t+1}$ and $H_{GX,t+1}$ are the portfolio volatility estimates from EWMA and GARCH-XCC processes respectively.¹⁵ In this study, we implement both GARCH methods for the estimation of portfolio VaR (LVaR) and report the corresponding results separately in the last section.

2.5. Incorporating Liquidity Dynamics into VaR

It is important to note here that, prior to LVaR predictions we empirically test the relationship between daily returns and daily VNET volatilities ($MDU_{i,t}$), as well as the relationship between daily returns and weighted relative spreads, by using in-sample logistic regression, with daily log returns (r_t) as binary respond variable (negative vs. positive). Results show that the average predicted probability of observing a positive return is a negatively sloped convex function of $MDU_{i,t}$, diminishing exponentially as $MDU_{i,t}$ increases (Figure 1). Similarly, for the weighted relative spreads SPR_{it} we find that more likely that SPR_{it} remains low (high), higher the probability of observing positive (negative) r_t for any given day t (Figure 2).¹⁶

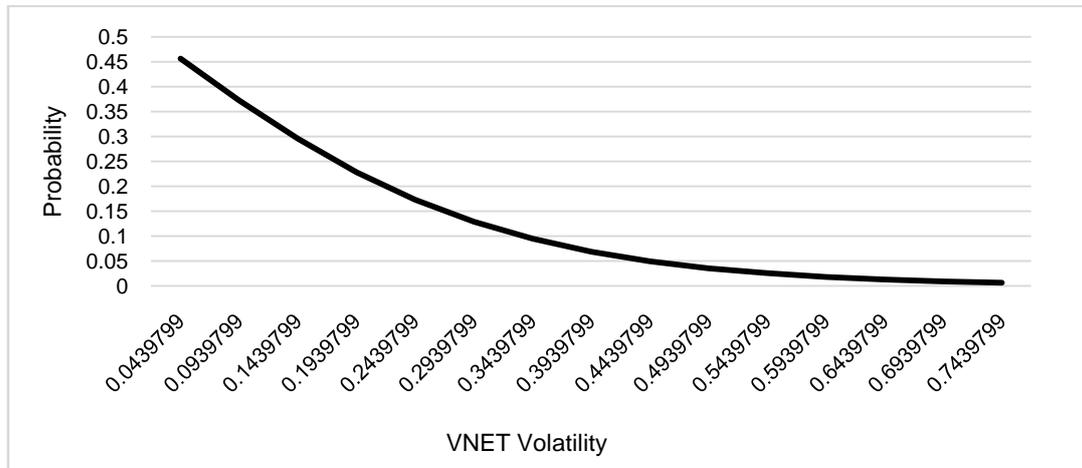


Figure 1. Average Probability of Observing Positive Returns vs. Daily VNET

¹⁴ ζ and ξ are the constant parameters with $\zeta > 0, \xi > 0$ and $\zeta + \xi < 1$ that allows for positive definiteness.

¹⁵ X standards for "C" ("D") if the model is CCC (DCC).

¹⁶ Related results are subject to another study and available upon request.

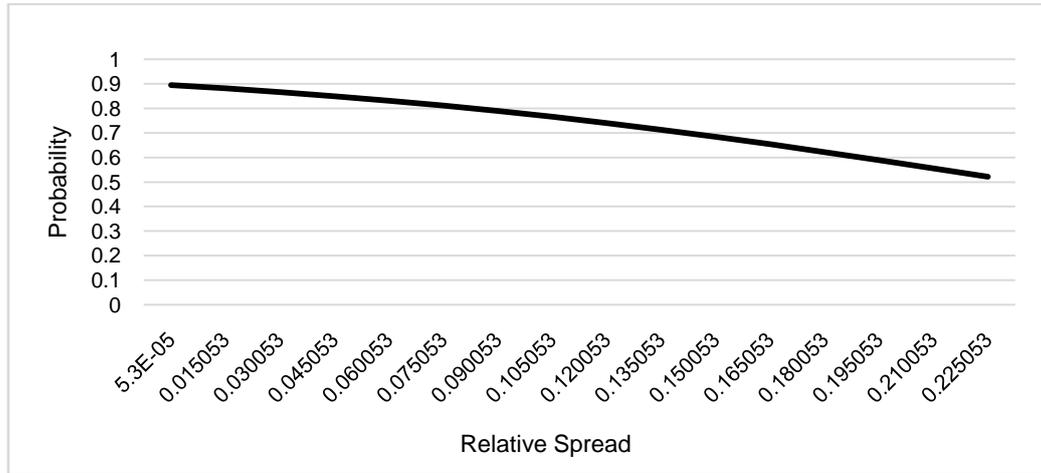


Figure 2. Average Probability of Observing Positive Returns vs. Daily Weighted Relative Spread

2.5.1. The Model for the Uncertainty in Market Depth

As mentioned previously, we are interested in the daily intraday level of depth uncertainty ($MDU_{i,t}$) and its interday variation (ϕ_t) through time. For a portfolio of assets ϕ_t can be defined as follows:

$$\phi_t = w_t' \begin{bmatrix} \sigma_{MDU_{i,t}}^2 & \cdots & \sigma_{MDU_{iN,t}} \\ \vdots & \ddots & \vdots \\ \sigma_{MDU_{Ni,t}} & \cdots & \sigma_{MDU_{NN,t}}^2 \end{bmatrix} w_t \quad (14)$$

The model focuses on the estimation of variance $H_{E,t}$ and $H_{GX,t}$ by using the variation in $MDU_{i,t}$ as the predictor. In this manner, we take the first k observations of ϕ_t and $H_{E,t}$ ($H_{GX,t}$) and by holding the portfolio weight vector w_t as constant, we run the estimation on a rolling window basis with 1 day increment.¹⁷ More precisely, by considering the EWMA volatility model, firstly we set the first k pairs of ϕ_t and estimate $H_{E,t}$ as $(\phi_{t-k}, H_{E,t-k}), (\phi_{t-k}, H_{E,t-k+1}), \dots, (\phi_t, H_{E,t})$. Secondly we define our model function as

$$\hat{H}_{E,t} = f(\phi_t, \hat{\beta}) = \hat{\beta}_{0,t} \exp(\hat{\beta}_{1,t} \phi_t) \quad (15)$$

$$s.t. \hat{H}_{EWMA,t} \geq H_{EWMA,t}$$

Here, \hat{H}_t is the liquidity adjusted return volatility for time t and $\hat{\beta}$ is the vector of coefficients, which best fits the given data by minimizing the sum of squares (SQ), such that $\frac{\partial SQ}{\partial \hat{\beta}} = 0$. Final values of $\hat{\beta}$ (ie. $\hat{\beta}_{0,t}$ and $\hat{\beta}_{1,t}$) are obtained through iterative approximation.¹⁸ Lastly, we rewrite Eq.16 as follows:

$$\tilde{H}_{EWMA,t+1} = \lambda \hat{H}_{EWMA,t} + (1 - \lambda)(R_t' R_t) \quad (16)$$

¹⁷ Although k can be of any arbitrary length, for the sake of simplicity and statistical robustness we take 100 days of rolling window and we left the determination of the optimal k value as a subject for future studies.

¹⁸ i.e. If M is the iteration number, then we get $\hat{\beta}_{l,t} \approx \hat{\beta}_{l,t}^{M+1} = \hat{\beta}_{l,t}^M + \Delta \hat{\beta}_{l,t}$, whereby $\Delta \hat{\beta}_{l,t}$ is the increment vector ($l = 1, 2$)

We assume that portfolio variance is a function of the variation in the uncertainty level of market depth (Eq. 15), whereby at each time instance t , we predict the liquidity adjusted portfolio variance $\tilde{H}_{EWMA,t+1}$ from historical $(\phi_t, H_{EWMA,t})$ pairs, giving the best historical fit. Besides, as seen in Eq. 15, we impose a lower boundary, which is critical to maintain the positiveness of the cost of liquidity by using the conventional return variance $H_{EWMA,t}$ as the control variate.¹⁹ Finally, the estimation of one day ahead $LVaR_{t+1}$ is straightforward to implement after determining $\tilde{H}_{EWMA,t+1}$.²⁰

$$LVaR_{E,t+1}^{MDU} = z_\alpha \sqrt{\tilde{H}_{E,t+1}} \quad (17)$$

2.5.2. The Model for Tightness

In order to incorporate the tightness dimension of liquidity into VaR, we use the first two moments of SPR_{it} , as proposed by (Bangia *et al.* 1999). The covariance matrix of SPR_{it} for a portfolio of assets can be defined as:

$$\psi_t = w_t' \begin{bmatrix} \sigma_{SPR_{i,t}}^2 & \cdots & \sigma_{SPR_{iN,t}} \\ \vdots & \ddots & \vdots \\ \sigma_{SPR_{Ni,t}} & \cdots & \sigma_{SPR_{NN,t}}^2 \end{bmatrix} w_t \quad (18)$$

Next, we define the mean vector of SPR_{it} for day t as:

$$\theta_t = [\theta_{1t} \quad \cdots \quad \theta_{Nt}] \quad (19)$$

We also use EWMA and GARCH-XCC for the volatility process underlying ψ_t , respectively. Last step is to incorporate ψ_t and θ_t to conventional VaR metric to obtain $LVaR$. For EWMA volatility model we have the following interpretation^{21,22}

$$LVaR_{E,t+1}^{SPR} = VaR_{E,t+1,\alpha} + 1/2 \left[w_t \theta_t + z_\alpha \sqrt{\lambda \psi_t + (1 - \lambda) R_t' R_t} \right] \quad (20)$$

Note that, we take $LVaR_{E,t}^{SPR}$ and $LVaR_{GX,t}^{SPR}$ as benchmark for LVaR models, in order to compare different performance metrics for the alternative models we introduce.

2.5.3. The Composite Model

Obviously, the most conservative approach is nested in the last LVaR model, which combines the latter two and aims to incorporate both the variation in depth uncertainty level and tightness dimension into VaR framework. Accordingly, for EWMA and GARCH-XCC volatility models we get the following composite structure:

$$LVaR_{E,t}^{CMP} = z_\alpha \sqrt{\tilde{H}_{E,t+1}} + 1/2 \left[w_t \theta_t + z_\alpha \sqrt{\lambda \psi_t + (1 - \lambda) R_t' R_t} \right] \quad (21)$$

¹⁹ Although it exists in previous studies (Lawrence and Robinson, 1995), we assume that cost of liquidity can not be negative.

²⁰ The estimation procedure for \tilde{H}_{t+1} and $LVaR_{t+1}$ is analogous for GARCH-XCC models.

²¹ It is important to note that we hold on to same simplifying assumption of Bangi *et al.* (1999), whereby we assume that worst spread occurs with the worst return.

²² The estimation procedure for LVaR is analogous for GARCH-XCC models.

2.6. Model Performance Evaluation

2.6.1. Regulatory Performance

From the perspective of regulatory authorities, for each model we focus on the number of violations, measured by the frequency of days when the realized return (loss) of portfolio on day t exceeds the estimated VaR_t ($LVaR_t$). We follow the Basel Criteria (II and III) and by running a back testing procedure on a daily rolling window basis with 250 days of historical in sample length we check at each time instance t , whether the ratio of total number of violations to in sample length (probability of failure: pf) exceeds the critical α level.²³ Institutions exceeding this critical level are exposed to higher market capital requirements as long as the ratio stays above α . In Basel Framework, the violation metric is divided into 3-color level, *a.k.a* *Basel Traffic Lights*, whereby the green zone reflects the fact that estimation failure is acceptable, the orange zone means that the failure is near the limit and finally the red zone implies that the allowed tolerance level is exceeded.²⁴ Under these criteria, we evaluate the regulatory performance for each model and at each time instance t in two dimensions; firstly we count the total number of violations and compare with the total out of sample observations (908 days) whether $\alpha \geq pf$ and secondly we count the days spent in each of the aforementioned colored zones. Therefore, we select those models with lower pf and less time spent in red zone as superior models.

2.6.2. Coverage

Performance evaluation with respect to statistical coverage rests on the main assumption that binary outcome variables (1 if violation, 0 else) of estimation methods are impossible to predict. Otherwise we would have a perfect model without any failure. Coverage tests are performed with respect to two widely accepted VaR validation methods known as the test for Unconditional Coverage (Kupiec, 1995) and the test for Conditional Coverage (Christoffersen, 1998): Unconditional Coverage Test (UC), tests for the null hypothesis stating that $\alpha \approx pf$.

According to UC, if the p-value exceeds that of the χ_v^2 , then the model is rejected.²⁵ Other than that, if violations occur successively, or cluster on any time interval, then UC test would be less reliable, since successive violations imply dependence. Therefore, such models should be rejected on the $v\%$ confidence level. Conditional Coverage test (CC) is designed for this particular reason and rests on the simultaneous implementation of violation independence and UC.²⁶ Another shortcoming of UC is that it focuses on the difference between the realized loss and the estimated value, by ignoring the existence of violations.²⁷ In order to overcome this problem and to be in line with the regulatory compliance we allocate each test outcome under four categories:

1. Rejected Underestimation (RU): The model has $\alpha < pf$ and VaR (LVaR) is underestimated (True).
2. Accepted Overestimation (AO): The model has $\alpha \geq pf$ and VaR (LVaR) is overestimated (True).
3. Rejected Overestimation (RO): The model has $\alpha \geq pf$ and VaR (LVaR) is overestimated (False).

²³ For instance for a historical back testing with sample length 250 the $\alpha = 0.01$ implies the number of exceedences must be at most $2.5 \approx 3$ days, otherwise the regulatory limit is violated.

²⁴ See Basel Framework for Minimum Capital Requirements (2016)

²⁵ χ_v^2 stands for the Chi-square distribution with v as the confidence interval. Throughout the coverage tests we take $v = 0.01$.

²⁶ See Kupiec (1995) and Christoffersen (1998) for details regarding test methodologies.

²⁷ Suppose there are 2 competing VaR models. For an hypothetical portfolio, the first model estimates a \$120 VaR, where as these cond estimates \$90. Further assume that the realized loss for $t+1$ is \$100. According to UC, model 1 is rejected and model 2 is accepted, although for model 1 $\alpha \geq pf$ and for model 2 $\alpha < pf$. The reason for rejection is that the size of the difference between estimated and realized values is higher for model 1 than that of model 2.

4. Accepted Underestimation (AU): The model has $\alpha < pf$ and VaR (LVaR) is underestimated (False).

Therefore, we consider only those models fall under categories RU and AO for the performance evaluation.

2.6.3. Predictive Quantile Loss

Predictive Quantile Loss (*PQL*) is a measure of average loss occurred on violation days (Koenker and Basset, 1978; Bertail et al. 2004; Komunjer 2004; Bao et al. 2006). *PQL* can be calculated as:

$$PQL_{\alpha} = \frac{1}{T} \sum_{t=1}^T [\alpha - I(r_t < \widehat{f}_{t,\alpha})] [r_t - \widehat{f}_{t,\alpha}] \quad (22)$$

Here, T is the total out of sample size, r_t is the realized return, $\widehat{f}_{t,\alpha}$ is the underlying VaR (LVaR) forecast model and $I(\cdot)$ is the indicator function. According to our performance evaluation criteria, the model with the smallest PQL_{α} is selected as the superior model. Moreover, we analyze the cost and benefit of using LVaR models by calculating the relative *PQL* against VaR models with the same underlying volatility structure as:

$$RPQL_{\alpha} = \frac{PQL_{\alpha}^{VaR} - PQL_{\alpha}^{LVaR}}{PQL_{\alpha}^{LVaR}} \quad (23)$$

2.6.4. Relative Cost of Liquidity

The last performance measure is the relative cost of liquidity (RCL) against the conventional VaR method, which is then compared with PQL_{α} to obtain a meaningful inference about the cost and benefit of the underlying LVaR model. RCL can be formulated as follows:

$$RCL_{\alpha} = \frac{1}{T} \sum_{t=1}^T \frac{\widehat{f}_{LVaRt,\alpha}}{\widehat{f}_{VaRt,\alpha}} \quad (24)$$

Here $\widehat{f}_{LVaRt,\alpha}$ and $\widehat{f}_{VaRt,\alpha}$ stand for the estimations obtained from the underlying LVaR and VaR models with given α level, respectively. According to this criterion, the model with the lowest RCL_{α} is preferred.

3. Theoretical Background

Related model studies in the field of LVaR are mainly clustered around two main classes, namely exogenous and endogenous models. The first exogenous approach is proposed by Bangia et al. (1999) and based on the tightness dimension of liquidity as measured by bid-ask spreads. A similar model by Ernst et al. (2012), modified the assumption regarding spread distribution by Cornish-Fisher expansion which improved the accuracy of forecasting. Exogenous models assume that no single investor could have any impact on the current market price. Thus, with this simplifying assumption the liquidity risk can be calibrated and incorporated into VaR by using the first two moments of the bid-ask spread. On the other hand, endogenous models generally assume that each individual investor has potential impact on the prevailing market price depending on her/his underlying position size. These models focus on variety of parameters including the optimal time to portfolio liquidation (Lawrence and Robinson, 1997; Jarrow and Subramanian, 1997; Almgren and Chriss, 2001; Almgren, 2003), market price response (Berkowitz, 2000), volume weighted spreads (Francois-Heude and van Wynendaele, 2001; Le Saout, 2002; Giot and Grammig, 2002; Qi and Wing, 2009) and duration between transactions (Emna and Chokri, 2014).

Although endogenous models are more realistic in scope, they focus mainly on intraday short term horizon for the estimation of VaR and majority of such models are also hard to

implement in all day practice. As already mentioned, each model relies on different variable explaining the cost of liquidity. Among them the most widely used indicator is the bid ask spread (weighted or not), representing the level of tightness. However, according to Engle and Lange (2001), this variable would only reflect the liquidity faced by small investors, by leaving high volume traders aside. Accordingly, they proposed an alternative measure of market liquidity VNET, defined as the difference between the number of shares bought and sold during any price duration. According to Bauwens and Giot (2001), it is possible obtain an intuitive measure of liquidity dynamics by computing the change in VNET over time. Rouetbi and Mamoghli (2014) investigated the dynamics of VNET and found a positive relationship between the volume and the volatility of VNET, which is also in line with the previous finding supporting the positive impact of depth on volatility (Naes and Skeltrpo, 2004).

As suggested by Persaud (2003) and McCoy (2003), traders, bankers and investors give more weight to variability in liquidity and hence not the average level of liquidity. By agreeing on the same fact, in this study we propose two alternative models for interday LVaR. The first model is a direct estimation of the liquidity adjusted return volatility by using VNET volatility dynamics. The main idea behind this model is similar to that of Berkowitz (2000), that each time instance we predict the additional variance incurred by liquidity risk. The second model has a composite structure, taking the VNET volatility dynamics and the first two moments of weighted bid ask spreads as an input to the exogenous model of Bangia *et al.* (1999). Being the composite model the superior, both models are easy in terms in terms of interpretation and found to be strong alternatives for tightness based LVaR models.

4. Results and Discussion

The first performance evaluation metric is based on the regulatory compliance, and we test for the case $\alpha = 0.01$.²⁸ According to results based on the number of violations and corresponding *pf* levels, the best models are those with the composite structure ($LVaR_{E,t}^{CMP}, LVaR_{GD,t}^{CMP}$), which then, regardless of the underlying volatility structure, followed by models based on tightness and depth uncertainty, respectively (Table 2). Considering the overall performance of underlying volatility models in regulatory test, GARCH-CCC is inferior to EWMA and GARCH-DCC models.

Table 2. Regulatory Test - Portfolio

Model	G	Y	R	V	pf
$LVaR_{E,t}^{CMP}$	776	132	0	11	0.009507
$LVaR_{GD,t}^{CMP}$	726	182	0	13	0.011236
$LVaR_{E,t}^{SPR}$	742	166	0	15	0.012965
$LVaR_{E,t}^{MDU}$	696	212	0	15	0.012965
$LVaR_{GD,t}^{SPR}$	555	353	0	17	0.014693
$LVaR_{GD,t}^{MDU}$	454	454	0	17	0.014693
$LVaR_{GC,t}^{CMP}$	551	357	0	18	0.015557
$VaR_{E,t}$	363	545	0	20	0.017286
$LVaR_{GC,t}^{MDU}$	427	481	0	20	0.017286
$VaR_{GD,t}$	410	498	0	21	0.01815
$LVaR_{GC,t}^{SPR}$	230	632	46	23	0.019879
$VaR_{GC,t}$	230	632	46	23	0.019879

Notes: $\alpha=0.01$. Y: number of days in "Yellow" category. R: number of days in "Red" category. pf: Probability of failure. V: Total Number of violations. G: number of days in "Green" category

²⁸ Regulatory rules impose only the use of $\alpha = 0.01$ for VaR models.

Table 3. Coverage Tests (Portfolio): Likelihood Ratio Test for Conditional Coverage (CC)

	CC		CC	
	$\alpha = 0.01$	Category	$\alpha = 0.05$	Category
$LVaR_{GD,t}^{CMP}$	2.427878	AU	18.760554	RO
$LVaR_{E,t}^{CMP}$	2.905679	AO	20.899538	RO
$LVaR_{E,t}^{MDU}$	7.113127	AU	12.800752	AO
$LVaR_{GD,t}^{MDU}$	7.451633	AU	13.50444	AO
$LVaR_{GC,t}^{CMP}$	7.858503	AU	13.5683	AO
$LVaR_{GC,t}^{MDU}$	9.096466	AU	7.080816	AO
$LVaR_{GD,t}^{SPR}$	12.626881	AU	6.397297	AO
$LVaR_{E,t}^{SPR}$	12.889425	AU	2.658501	AU
$VaR_{E,t}$	13.508567	AU	5.357317	AU
$VaR_{GC,t}$	15.664078	RU	3.706196	AU
$LVaR_{GC,t}^{SPR}$	15.664078	RU	5.099647	AO
$VaR_{GD,t}$	19.186978	AU	6.303789	AU

Notes: AU: Accepted Underestimation, AO: Accepted Overestimation, RU: Rejected Underestimation. RO: Rejected Overestimation

Besides, regarding the CC test, all LVaR Models, regardless of the α level, perform overall better than the conventional counterparts (Table 3). At level $\alpha = 0.01$, $LVaR_{E,t}^{CMP}$ is the superior model falling under category AO, whereas $VaR_{GC,t}$ and $LVaR_{GC,t}^{SPR}$ are the worst performing models (RU), caused by the GARCH-CCC volatility assumption.²⁹ Corresponding results related to $\alpha = 0.05$ indicate that $LVaR_{GC,t}^{SPR}$ is the best model followed by $LVaR_{GD,t}^{SPR}$, $LVaR_{GC,t}^{MDU}$, $LVaR_{GD,t}^{MDU}$, $LVaR_{E,t}^{MDU}$ and $LVaR_{E,t}^{CMP}$ all falling under category AO.

Moreover, the PQL test indicates that all LVaR models, in particular those following the EWMA and GARCH-DCC volatilities, exhibit lower loss on violation days on average, when compared to that of conventional VaR models with varying volatility structures (Table 4).

According to PQL results related to $\alpha = 0.01$ level, $LVaR_{E,t}^{CMP}$ is again the superior model, followed by $LVaR_{GD,t}^{CMP}$, $LVaR_{E,t}^{SPR}$, $LVaR_{E,t}^{MDU}$ and $LVaR_{GD,t}^{MDU}$, respectively. At level $\alpha = 0.05$, the composite type models ($LVaR_{E,t}^{CMP}$, $LVaR_{GD,t}^{CMP}$) and models based solely on depth uncertainty ($LVaR_{E,t}^{MDU}$ and $LVaR_{GD,t}^{MDU}$) found to be superior among other models.

Table 4. Average Predictive Quantile Loss across Models (Portfolio)

Model	$\alpha=0.01$	$\alpha=0.05$
$LVaR_{E,t}^{CMP}$	2.127E-04	4.627E-04
$LVaR_{GD,t}^{CMP}$	2.295E-04	4.896E-04
$LVaR_{E,t}^{SPR}$	2.404E-04	5.908E-04
$LVaR_{E,t}^{MDU}$	2.446E-04	5.248E-04
$LVaR_{GD,t}^{MDU}$	2.553E-04	5.442E-04
$LVaR_{GC,t}^{CMP}$	2.616E-04	5.654E-04
$LVaR_{GD,t}^{SPR}$	2.669E-04	5.820E-04
$VaR_{E,t}$	2.789E-04	6.968E-04
$LVaR_{GC,t}^{MDU}$	2.944E-04	6.248E-04
$VaR_{GD,t}$	3.068E-04	6.667E-04
$LVaR_{GC,t}^{SPR}$	3.075E-04	6.837E-04
$VaR_{GC,t}$	3.498E-04	7.631E-04

²⁹ Since CC test is a combination of UC and independence test, in order to save space were present only the results corresponding to CC together with our control categories for model acceptance.

Finally, *RCL* and *RPQL* results imposed by different *LVaR* models (Table 5) indicates that that the average *RCL* ranges between 6% and 22% against *RPQL* varying between 13.75% and 34%, across *LVaR* models with differing volatility dynamics.

Table 5. Relative Cost of Liquidity vs. Relative PQL ($\alpha=0.01$) –Portfolio

Model	RCL	RPQL
$LVaR_{E,t}^{SPR}$	0.063142	0.159885
$LVaR_{GC,t}^{SPR}$	0.062475	0.137524
$LVaR_{GD,t}^{SPR}$	0.062758	0.149400
$LVaR_{E,t}^{MDU}$	0.154083	0.165516
$LVaR_{GC,t}^{MDU}$	0.127786	0.188136
$LVaR_{GD,t}^{MDU}$	0.132147	0.201958
$LVaR_{E,t}^{CMP}$	0.216801	0.310917
$LVaR_{GC,t}^{CMP}$	0.182092	0.336841
$LVaR_{GD,t}^{CMP}$	0.184984	0.337109

4. Conclusion

In line with the previous studies we emphasize the use of *LVaR* models that are able to incorporate the cost of liquidity dynamics in to conventional *VaR* metrics. We propose two alternative approaches by setting the tightness based *LVaR* approaches a control benchmark throughout the evaluation process. Our first model is based on the interday volatility dynamics of *VNET*, which is introduced by Engle and Lange (2001) as an intraday indicator for the dynamics of market depth. The second model incorporates the tightness dimension to the latter by using the methodology of Bangia *et al.* (1999). Our model evaluation metric considers both regulatory as well as statistical viewpoints. We firstly report that all *VaR* models perform strictly poor, especially from the regulatory perspective. *LVaR* models based on *VNET* found to be a strong alternative to existing tightness (spread) based *LVaR* models. We report the composite model, as the superior model. More importantly, from the overall perspective, the cost and benefit measured by *RCL* and *RPQL* indicate the efficiency of *LVaR* models. Similar to that of Stange and Kaserer (2008), our general finding is that intraday data should be incorporated into risk management process, since models based on larger data set perform better in forecasting.

References

- Alexander, C., 2008. *Market risk analysis, volume I: Quantitative methods in finance*. Chichester: John Wiley & Sons.
- Almgren, R. and Chriss, N., 2001. Optimal execution of portfolio transactions. *Journal of Risk*, 3(1), pp.5-39. <https://doi.org/10.21314/JOR.2001.041>
- Almgren, R., 2003. Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematical Finance*, 10(1), pp.1-18. <https://doi.org/10.1080/135048602100056>
- Bangia, A., Diebold, F.X., Schuermann, T., and Stroughair, J.D., 1999. Modeling liquidity risk with implications for traditional market risk measurement and management. University of Pennsylvania *Center for Financial Institutions Working Papers*, No. 99-06, pp.1-16.
- Bao, Y., Lee, T.H., and Saltoglu, B., 2006. Evaluating predictive performance of value-at-risk models in emerging markets: a reality check. *Journal of Forecasting*, 25(2), pp.101-128. <https://doi.org/10.1002/for.977>
- Basel Committee on Banking Supervision, 2016. *Minimum capital requirements for market risk* [online]. Available at: <<http://www.bis.org/bcbps/publ/d352.htm>> [Accessed on 1 June 2016].

- Bauwens, L. and Giot, P., 2001. Econometric modelling of stock market intraday activity. In: B.H. Baltagi, Y. Hong, G. Koop, and W. Kramer eds. 2001. *Advanced studies in theoretical and applied econometrics*. Boston: Kluwer Academic Publishers. <https://doi.org/10.1007/978-1-4757-3381-5>
- Berkowitz, J. 2000. Breaking the silence, liquidity risk. *Risk Magazine*, 13, pp.105–108.
- Bertail, P., Haefke, C., Politis, D.N., and White, H., 2004. A subsampling approach to estimating the distribution of diverging statistics with applications to assessing financial market risks. *University of California at San Diego, Economics Working Paper Series*.
- Chakrabarty, B., Moulton, P.C., and Shkilko, A., 2012. Short Sales, Long Sales, and the Lee-Ready Trade Classification Algorithm Revisited. Cornell University, School of Hospitality Administration [online]. Available at: <<http://scholarship.sha.cornell.edu/articles/2/>> [Accessed on 1 June 2016].
- Christoffersen, P., 1998. Evaluating interval forecasts. *International Economic Review*, 39(4), pp. 841-862. <https://doi.org/10.2307/2527341>
- Rouetbi, E. and Mamoghli, C., 2014. Measuring liquidity in an emerging market: The Tunis stock exchange. *International Journal of Economic and Financial Issues*, 4(4), pp.920-929.
- Engle, R. F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, 50(4), pp.987–1007. <https://doi.org/10.2307/1912773>
- Engle, R.F. and Lange, J., 2001. Predicting VNET: A model of the dynamics of market depth. *Journal of Financial Markets*, 4(2), pp.113-142. [https://doi.org/10.1016/S1386-4181\(00\)00019-7](https://doi.org/10.1016/S1386-4181(00)00019-7)
- Engle, R., 2002. Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business and Economic Statistics*, 20(3), pp.339–350. <https://doi.org/10.1198/073500102288618487>
- Ernst, C., Stange, S., and Kaserer, C., 2012. *Measuring market liquidity risk - which model works best*. CEFS Working Paper Series. No.2012-01.
- Francois-Heude, A. and P. van Wynendaele, 2001. *Integrating liquidity risk in a parametric intraday VaR framework*. [online] Available at: <https://www.researchgate.net/publication/244097447_Integrating_Liquidity_Risk_in_a_Parametric_Intraday_VaR_Framework> [Accessed on 1 June 2016].
- Giot, P. and Grammig, J., 2002. How large is liquidity risk in an automated auction market? *CORE Discussion Papers*, No. 2002054, pp.1-31.
- Jarrow, R. and Subramanian, A., 1997. Mopping up liquidity. *Risk Magazine*, pp. 170-173.
- Jorion, P., 2006. *Value at risk: The new benchmark for managing financial risk*. 3rd ed. New York, NY: McGraw-Hill.
- Koenker, R. and Bassett, G., 1978. Regression quantiles. *Econometrica*, 46(1), pp. 33-50. <https://doi.org/10.2307/1913643>
- Komunjer, I., 2004. Quasi-maximum likelihood estimation for conditional quantiles. *Journal of Econometrics*, 128(1), pp.137-164. <https://doi.org/10.1016/j.jeconom.2004.08.010>
- Kupiec, P., 1995. Techniques for verifying the accuracy of risk management models. *Journal of Derivatives*, 3(2), pp.73-84. <https://doi.org/10.3905/jod.1995.407942>
- Lawrence, C. and Robinson, G., 1997. Liquidity, dynamic hedging and value at risk. *Risk Management for Financial Institutions*, pp.63-72.
- Lee, C. and Ready, M., 1991. Inferring trade direction from intraday data. *Journal of Finance*, 46(2), pp.733-46. <https://doi.org/10.1111/j.1540-6261.1991.tb02683.x>
- McCoy, C., 2003. Liquidity black holes: Testing the theory's predictions. In: A., Persaud, ed. 2003. *Liquidity black holes: Understanding, quantifying and managing financial liquidity risk, vol I*. London: Haymarket House. pp. 139-151.
- Le Saout, E., 2002. Incorporating liquidity risk in VaR models. *Paris 1 University, Working Paper*.
- Naes, R. and Skjeltorp, J., 2004. Order book characteristics and the volume-volatility relation: Empirical evidence from a limit order market. *EFA 2004 Maastricht Meetings Paper*, No. 2742.

- O'Hara, M., Ellis, K., and Michaely, R., 2000. The accuracy of trade classification rules: Evidence from Nasdaq. *Journal of Financial and Quantitative Analysis*, 35(4), pp.529-551. <https://doi.org/10.2307/2676254>
- Persaud, A., 2003. *Liquidity black holes: Understanding, quantifying and managing financial liquidity risk*. London: Risk.
- Qi, J. and Ng, Wing Lon, 2009. Liquidity adjusted intraday value at risk. *Proceedings of the World Congress on Engineering 2009, WCE*, July 1 - 3, 2009. London: IAENG.
- Risk Metrics, 1996. *Risk metrics technical document*. [online] Available at: <<https://www.msci.com/documents/10199/5915b101-4206-4ba0-ae2-3449d5c7e95a>> [Accessed on 1 June 2016].
- Stange, S. and Kaserer, C., 2008. The impact of order size on stock liquidity - A representative study. *CEFS Working Paper*, No. 2008-9.
- Tse, Y. and Tsui, A. K. C., 2002. A multivariate generalized autoregressive conditional heteroscedasticity model with time-varying correlations. *Journal of Business & Economic Statistics*, 20(3), pp.351-362. <https://doi.org/10.1198/073500102288618496>