Eurasian Journal of Economics and Finance, 7(2), 2019, 32-41
DOI: 10.15604/ejef.2019.07.02.003

EURASIAN JOURNAL OF ECONOMICS AND FINANCE

www.eurasianpublications.com

EXAMINING THE APPLICATION OF MATHEMATICS IN ECONOMICS*

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Abstract

In most scientific disciplines, mathematical models work well empirically and experimentally. However, it is widely recognized that in standard (i.e. neoclassical) economics, mathematical models do not work quite as well, not even at the theoretical level. The question is why. This diagnostic study, which is one of the first in its kind, addresses this question and reveals that mathematics is non-rigorously applied in economics by focusing on the case of ordinal utility theory (which rests at the foundation of neoclassical economics), and identifying three mathematical oversights there. For these reasons, this paper calls for a reconstruction of economic theory with a much more solid foundation than that of the currently dominant paradigm, and with a much more rigorous application of logic and mathematics.

Keywords: Hidden Mathematical Contradictions, Ordinal Utility, General Equilibrium

JEL Classifications: C02, D01, D50

1. Introduction

Galileo Galilei (1564-1642) is famous for his perception that regularities of Nature have mathematical forms (Drake, 1957). Given the widely recognized theoretical, empirical and experimental failures of many mathematical models in economics (e.g. Arrow, 1986; Bouchaud, 2008; Thaler, 2016), one may doubt the validity of the extension of this view to economics as Hudson (2000) does. Alternatively, one may question the accuracy of the application of mathematical concepts employed in these models. The latter accuracy is rarely examined in the economics literature. However, this paper does so and proves that it is the incorrect application of mathematics that is the root cause of the failure of these models. We demonstrate this point by focusing on ordinal utility theory, which Vilfredo Pareto (1848-1923) pioneered (Aspers, 2003) and rests at the foundation of neoclassical economics, and which far from explaining the mysteries of the real-world economy, has generated many theoretical, empirical and experimental puzzles. An example of the theoretical puzzles can be seen in the works of leading neoclassical economists such as Sonnenschein (1972, 1973), Mantel (1974), Debreu (1974); and Mas-Colell *et al.* (1995) who note that the central models of neoclassical economics lead to "anything goes" status. On the other hand, there is virtually no empirical evidence for the existence of any ordinal utility function. Moreover, experimental economics demonstrates that ordinal utility theory fails to work well. For example, experiments on endowment effect, loss aversion, and status quo bias (Kahneman *et al.* 1991) or context dependency (Tversky and Simonson, 1993) cast doubt on existence of ordinal utility functions.

* The author gratefully appreciates the comments received from the Editorial Board. The author wishes to dedicate this paper to the memory of his dearest brother Reza.

Clearly, any oversight such as a hidden internal contradiction in the proof of the existence of ordinal utility functions can explain the emergence of all the foregoing problems, as one cannot have external consistency from an internally inconsistent theory. In this paper, we prove that the root of these problems lies at the lack of rigor (see Appendix 1) by the standards of contemporary mathematics in the application of the latter discipline. This includes recent proofs of the existence of ordinal utility functions (Voorneveld and Weibull, 2016). The highlights of this research are the following points:

- Identification of the reasons why mathematical models of neoclassical economics, instead of explaining the mysteries of the real-world economy, generate many theoretical, empirical and experimental puzzles of their own.
- Proof that the axiom of reflexivity in ordinal utility theory violates the axiom of scarcity of resources, leading to the explanation of the already recognized "free lunches" in general equilibrium models of Arrow and Debreu (1954) and Samuelson (1958).
- Acknowledgement that each good can have more than one use or application, and hence an individual can have more than one preference-ordering of a set of goods, and proof that this leads to an internal inconsistency for ordinal utility theory.
- Noting that the mathematical concept of function is misapplied in ordinal utility theory, and demonstrating that this is the reason why general equilibrium models lead to "anything goes" status.
- Calling for a reconstruction of economic theory with a more solid foundation than that of the standard paradigm, such that the key characteristics of the real-world economy and its market participants are defined more fittingly, and logic and mathematics are applied more rigorously.

The structure of the rest of this paper is as follows. Sections 2, 3 and 4 identify three different mathematical oversights in ordinal utility theory. Section 5 explains the implication of these results for general equilibrium theory and Section 6 concludes.

2. Inconsistency of the axiom of reflexivity with the axiom of scarcity of resources

In ordinal utility theory (Jehle and Reny, 2011), under the axiom of reflexivity, which is implied by the axiom of completeness[1], an individual can replace a good with its perfect substitute in a *timeless* transaction with no change in his/her welfare. Moreover, he/she can do so infinitely many times timelessly with no change in his/her welfare. In other words, ordinal utility theory assumes that the exchange of perfect substitutes between individuals need not take anytime, even if that is carried out infinitely many times. We prove in Proposition 1 that this assumption contradicts the axiom of scarcity of resources.

Proposition 1: *Market participants can always obtain a free lunch/money pump within a finite time in any model of a competitive, efficient and frictionless economy of the neoclassical economics as long as one can replace perfect substitutes timelessly.*

Proof: Consider a private market participant who borrows from another private market participant a loan at the start of time-period θ_{d-1}, and pays off this loan with its accrued interest at the end of time-period θ_{d-1}, which is also the start of time-period θ_d, for every date $d = 0,1,2,3,\ldots$ (of no length in time) such that at every date $d > 0$ the new loan equals all that is needed (in terms of principal and interest) to pay off the previous loan, hence the borrower never defaults. The money borrowed *and* repaid with interest can be *either* in the fiat form *or* commodity form (e.g. gold), but not a combination of these two. This is based on the assumption that at each

[1] Axioms of ordinal utility theory are explained in Jehle and Reny (2011), the most important of which are completeness and transitivity; these together with utility maximization hypothesis define "rational" economic behavior of individuals under certainty in neoclassical economics.

date $d > 0$ valid transactions e.g. repayment of the old loan and the borrowing of the new loan can take place concurrently without taking any time, which is permissible in the standard paradigm. (Note that we do not claim that this assumption can hold in real-life.)

Let us assume the initial loan is $l_0 > 0$ at date 0 and the total amount of the principal and interest payable is l_d at date d, and the amount re-borrowed at date $d < 0$ is also l_d. The new lender at each date does not need to be the same as the old one. Hence, the cash flows of the borrower in respect of these loans at dates $d = 0,1,2,3,$ ...and their repayments at dates $d = 0,1,2,3,$ ... as perceived in the standard paradigm are as in Table 1. In terms of record-keeping, the borrower records the inflows and outflows at each date $d = 0,1,2,3,$ ... as if they concur. Let us note that these equal inflows and outflows of money are either in the fiat form or commodity form (e.g. gold), but not both, and hence they are perfect substitutes at each date $d = 0,1,2,3,$ ....

Table 1. Cash flows of the borrower in respect of loans

Date	0	1	2	3	…….
Cash flow	l_0	$l_1 - l_1$	$l_2 - l_2$	$l_3 - l_3$	…….
OR: Net cash	l_0	0	0	0	…….

Thus, the borrower obtains a sure free lunch of l_0, which can be of any finite size. Remarkably, this process can take place not only over an infinite horizon, but also over a finite horizon $0 < H < \infty$. For example, the length of the duration of each successive loan starting at dates $d = 0,1,2,3,$ ... can be $\theta_d = 2^{-d-1}H$ where $\sum_{d=0} \theta_d = H$. In this case, at the end of H, the duration of the loan at this date is zero, which is equivalent to having had no loan. Hence, the loans are investments that are always repaid with their returns.

On the other hand, instead of engaging in repeated borrowing and repayment by re-borrowing during period H, one can do the same in every one of the periods θ_d where $d = 0,1,2,3,$ .... This leads to a free lunch of an infinite size by the end of H, hence breaching the axiom of scarcity. These free lunches/money pumps disappear as soon as one drops the assumption that replacement of perfect substitutes between parties can occur timelessly. For, this makes it impossible for the same market participant to re-borrow and repay the same amount of money at the same point of no length in time at dates $d = 0,1,2,3,$ ... in any model. This is why when borrowing and repayment by re-borrowing takes place repeatedly in real-life as in a Ponzi scheme, the borrower eventually defaults. Let us note that this Proposition holds even when H is of only a positive infinitesimal length of time, hence, the validity of this Proposition is not dependent on the elapse of any significant time.

3. Non-uniqueness of preference-orderings and its consequences

When all the goods in the same set are compared, preferences for using them can be evaluated for different purposes e.g. for consumption purposes, or for investment purposes, or for the purpose of trading in them, or for the purpose of throwing them away, or for a combination of these purposes, etc… In that case, there is no guarantee that all these different comparisons will generate the same preference-ordering of the same set of goods as implied by ordinal utility theory. For instance, the individual who prefers to consume apples rather than oranges may also prefer to trade in oranges than in apples as he/she finds trading in oranges to be more profitable than trading in apples. This leads to different preference-orderings of the same set of goods. However, ordinal utility theory makes no distinction between the different preference-orderings arising from the various applications of the same set of goods. In fact, it implicitly assumes that a "rational" individual has only one preference-ordering of a set of goods, irrespective of the fact that each good can be applied for different purposes, and the latter can generate different preference-orderings of the same set of goods (see Appendix 2). The failure to distinguish the different preference-orderings arising from the various applications of an individual's wealth in ordinal utility theory means that the individual is implicitly presumed to be indifferent between

these various applications of each good, whilst these different applications can have opposing impacts on his/her welfare, thus generating an internal contradiction, as Proposition 2 illustrates.

Proposition 2: *If an individual prefers the consumption of a positive quantity of a good that he/she owns to its free disposal (and thus losing the right to its consumption), then an ordinal utility function for his/her preferences cannot exist.*

Proof: Let us consider an individual with ordinal utility function u who initially owns $x > a < 0$ quantity of a specific good (for simplicity, the set of goods as objects of choice that he/she owns is taken to have only one element with quantity x) leading to a level of utility for him/her represented by $u(x)$, where u is a strictly increasing function. The individual has a choice between the consumption and the free disposal of the same quantity of the good. He/she knows that for him/her, consumption of the good is preferable to its disposal, and wishes to find out whether his/her utility function $u(x)$ will correctly reflect his/her preference-ordering. To test the validity of ordinal utility theory in this respect, the individual engages in a thought experiment, where the intention is to see if this theory can correctly distinguish the results of his/her actions on his/her welfare as measured by his/her utility function. The test is described in the *alternative* Scenarios I and II, explained in the next paragraph. By assumption there is no difference in either the characteristics of the wealth of the individual or his/her preferences in Scenarios I and II. Moreover, the available information, context, choice set, choice environment, and states of world in Scenarios I and II are the same by assumption. Furthermore, the duration of the events in each of the alternative Scenarios I or II is the same and equal to a positive infinitesimal such that if each event starts at date d_s, it will finish at date d_f.

In Scenario I, he/she consumes quantity a of the good; hence the change in his/her utility due to this event will be $u(x) - u(x - a)$. In the alternative Scenario II, he/she decides to throw it away, which he/she knows will lead to the loss of the same quantity a of the good and thus he/she loses the right to its consumption; hence the change in his/her utility due to this event will also be

$$u(x) - u(x - a).$$

It follows from the foregoing that the change in this individual's utility as a result of any one of these two events in Scenario I or II is exactly the same i.e. $u(x) - u(x - a)$, as his/her wealth and his/her utility at date d_s immediately before these events are the same in either Scenario I or II**,** and his/her wealth and his/her utility at date d_f immediately after these events are also the same in either Scenario I or II. However, he/she will not be indifferent between events in Scenarios I and II, contrary to the prediction of ordinal utility theory, as by assumption he/she prefers to consume the good to losing it, hence a contradiction for the representation of preferences by an ordinal utility function.

Let us note that the different impacts on the individual's welfare occur as a result of the different applications of the same good, which ordinal utility theory fails to recognize, as it implicitly assumes that all the different applications of the same good have the same welfare effect on the individual. Let us also note that by assumption there is no difference in either the characteristics of the wealth of the individual or his/her preferences on the consumption versus disposal of the good in the alternative Scenarios I or II. Moreover, the available information, context, choice set, choice environment, and states of world in Scenarios I and II are the same by assumption; hence the different impacts on the individual's welfare cannot be attributed to a change in any of these factors. In fact, the different welfare effects merely reflect the different purposes that the application of the same good is subject to in Scenarios I and II.

4. Misapplication of the mathematical concept of function

The definition of a function (Hamilton, 1982) requires that for a real-valued ordinal utility function for a bundle of goods to exist, it must be single-valued. However, if the bundle has only one good in it, with every fixed quantity x, the utility of this bundle, under ordinal utility theory, can be

represented by $u(x)$, the value of which can be 1, 2, 1.5, or indeed any real number, hence u will not be single-valued, and thus not a function. Similarly, if the bundle has more than one good, under ordinal utility theory, given every fixed vector $\mathbf{x}$ representing the fixed quantities of the goods in the bundle, and $u(\mathbf{x})$ representing the utility of the bundle as a whole, the value of $u(\mathbf{x})$ can be 1, 2, 1.5, or indeed any real number, hence u will not be single-valued, and thus not a function, in this case either, making utility maximization (subject to budgetary constraints) impossible.

5. Implications of the foregoing results for general equilibrium theory

It follows from Sections 2, 3 and 4 that the procedure in the existing literature (e.g., Jehle and Reny, 2011) for the derivation of an individual's demand function from maximizing a utility function subject to his/her budgetary constraint is fundamentally flawed. That is to say, no ordinal utility function (subject to budgetary constraints) is maximized despite the appearances of this procedure being applied. This is the reason why Arrow (1986, p. S388) finds that in the Arrow-Debreu model of general equilibrium: "the hypothesis of rational behavior has in general no implications". This leads to "anything goes" in this model, a status which Debreu (1991, p. 2-3) recognizes as being characteristic of internally inconsistent deductive structures: "A deductive structure that tolerates a contradiction does so under the penalty of being useless, since any statement can be derived flawlessly and immediately from that contradiction."

Appendix 3 explains that the free lunch in Proposition 1 of Section 2, which is an inherent characteristic of the competitive, efficient and frictionless economy of the standard paradigm, is the reason why general equilibrium models, such as those of Arrow and Debreu (1954) and Samuelson (1958) representing this economy, generate free lunches.

Falahati (2019a) draws on Proposition 1 and rigorously proves that the law of one price in its standard sense cannot hold in any competitive, efficient and frictionless economy. This undermines the validity of any equilibrium model of the economy, which has dominated neoclassical economics. It also helps resolve very many puzzles in microeconomics, finance and macroeconomics. Falahati (2019b) reveals that the economy has inherently a tendency to generate cyclical swings rather than any equilibrium.

6. Conclusion

In comparison to the achievements of most other scientific disciplines over the past 200 years, it is widely recognized that standard economic theory, in terms of its external consistency, has a great deal to improve upon. This paper highlights that it also suffers from internal inconsistency at its foundations. The history of scientific endeavors illustrates that certain fundamental ideas that have been dominant and spell-binding for many generations turn out to be void and unjustified for subsequent generations, as it is always possible for mankind to err either individually or collectively in his/her conception or perception of facts and logic, hence the provisional nature of science. We propose that what is now required is a serious revision of the foundations of standard economic theory to remove its internal inconsistencies as a pre-requisite to achieve external consistency. That can lead to a quantum leap in economics. Quantum leaps in intellectual disciplines arise from breaking the spells cast by previous generations of scientists, and setting in motion the trains of efforts and thoughts needed to reconstruct them. That is not easy, not only because conceptual flaws, unnoticed by previous generations of scholars, are inherently hard to discover, but also because new ideas often face strong resistance from believers of the challenged dominant old ideas. This can retard genuine breakthroughs. Nonetheless, all scientific disciplines can go through radical revisions and can eventually overcome obstacles, as biology, physics and mathematics have done. It is such exercises that have made pure mathematics the most rigorous intellectual discipline that humankind has developed. All sciences need such exercises continually. In the absence of doing so, one is most likely to remain unaware of any possible errors of previous generations of scientists, and one is bound to repeat their mistakes, thus science cannot make much progress. The first step in this regard in relation to standard economic theory is to scrutinize the foundational works of its pioneers and see if they have erred

anywhere, and if so where. This helps identify, avoid and correct any hidden oversights there. This paper presents the first step in this direction, where we prove that mathematics is incorrectly applied in economics by identifying three mathematical oversights, summarized in the following paragraph, in ordinal utility theory which rests at the foundation of neoclassical economics.

We prove that the axiom of reflexivity is inconsistent with the axiom of scarcity of resources. The misapplication of the mathematical concept of function in ordinal utility theory is also highlighted. Moreover, it is noted that the same good (or set of goods) can have different uses or applications, which can generate different preference-orderings and thus different welfare effects, which ordinal utility theory fails to recognize. For, ordinal utility theory implicitly assumes individuals are indifferent between any pair of the different applications of the same good (or set of goods). This leads to contradictions. For instance, an apple owner can either consume the apple or throw it away; each of these acts has the same effect on the wealth of the individual (represented by a reduction of his/her wealth in the input of his/her utility function), but a different impact on his/her welfare. However, the apple owner's ordinal utility function fails to distinguish the latter welfare differences, as it displays the same impact (represented by the same reduction in the output of his/her utility function); as if the individual were indifferent between these two acts. More generally, the diversity of the effects on an individual's welfare from the different applications of the same good cannot be picked up by any utility function. This makes it impossible to have a function represent an individual's preference-orderings of the various applications of the same good, and by extension of the same bundle of goods.

This paper explains the reasons why models of general equilibrium in economics lead to anything goes status and generate free lunches. These findings confirm the validity of the prediction of the leading neoclassical economist Hahn (1982, p. 373) on the fate of neoclassical economics that it "will not survive logical scrutiny". It is proposed in this paper that these findings underlie the theoretical, empirical and experimental failings of neoclassical economics. We therefore concur with Keynes (1933, p. 188) where he states: "How disappointing are the fruits, now that we have them, of the bright idea of reducing Economics to a mathematical application of the hedonistic calculus of Bentham."

It is therefore reasonable to expect that self-respecting economists, as scientists, should follow the example of great scientists such as Charles Darwin (1809-1882) who was prepared to correct his fundamental views when faced with the logical analysis of his new findings (Darwin 1860, p. 13): "I can entertain no doubt, after the most deliberate study and dispassionate judgement of which I am capable, that the view that most naturalists entertain, and which I formerly entertained ― namely, that each species has been independently created ― is erroneous." Likewise, it is now unscientific to bury one's head in the sand and ignore the overwhelming factual, logical and mathematical evidence against standard economic theory.

We conclude by stating what seems obvious to many outside the economics profession, and to many inside it but thinking outside the neoclassical paradigm: Economics needs a new and much more solid foundation than the currently dominant paradigm, such that it defines the key characteristics of the economy and its players much more aptly and precisely, with a much more rigorous application of logic and mathematics than applied hereto, such that it avoids any internal inconsistency; as a pre-requisite to achieve external consistency. To see how this can be done, readers may consult Falahati (2012, 2019a, 2019b), where a vast number of theoretical, empirical and behavioral puzzles in microeconomics, financial economics and macroeconomics are resolved.

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Appendix 1. Lack of rigor in standard economic theory

To cite one example, consider the Arrow and Debreu (1954) paper on the existence of equilibrium in a competitive economy which is of fundamental importance in neoclassical economics. According to Weintraub (2011, p. 208-9): "the Arrow-Debreu paper was not comprehensively refereed".

In particular, it is clear from the facts coming to light recently through the accounts of Weintraub (2011) and Düppe and Weintraub (2014) that no mathematician of the Nicolas Bourbaki school of thought, being dedicated to the advancement of rigor in modern mathematics since 1935, ever refereed it, and the Arrow-Debreu paper was published without heeding the advices of one of its two referees and the associate editor responsible for it. Our diagnostic study here is the first in scrutinizing the mathematical foundations on which the Arrow and Debreu model of general equilibrium is built and in identifying its hidden causes of internal inconsistencies from the perspective of modern mathematics, where mathematical concepts such as the *infinitesimal* and *function* are well-defined. From an economist's perspective, the most obvious inadequacy of the Arrow and Debreu model is its lack of room for: everyday transactions, fiat money and financial institutions (Hahn, 2005, p. 21-43) which generate, inter alia, the "Hahn problem". Rogers (2018) explains how this problem persists in dynamic stochastic general equilibrium models, which Central Banks have used with ill-fated consequences.

Appendix 2. Preference-ordering in ordinal utility theory

Sen (1977, p. 335-336) critiques this single all-purpose preference-ordering aspect of ordinal utility theory from a different angle: "A person is given *one* preference ordering …to reflect his interests, represent his welfare,… and describe his actual choices and behavior. Can one preference ordering do all these things? A person thus described may be 'rational' in the limited sense of revealing no inconsistencies in his choice behavior, but if he has no use for these distinctions between quite different concepts, he must be a bit of a fool. The *purely* economic man is indeed close to being a social moron. Economic theory has been much preoccupied with this rational fool decked in the glory of his *one* all-purpose preference ordering."

In Proposition 2 in Section 3, we prove that this assumption of ordinal utility theory suffers from hereto hidden internal inconsistencies, and thus this "fool" is not rational in a *logical* sense, and this is the reason why mathematical models built on this premise fail.

Appendix 3. Free lunches in general equilibrium models

Proposition 1 in Section 2 explains the well-known free lunch in the overlapping general equilibrium model of Samuelson (1958), as an example of a model of a competitive, efficient and frictionless economy in which similar repeated borrowing and re-borrowing take place. The Arrow-Debreu model of general equilibrium is another example of a model of a competitive, efficient and frictionless economy that relies on ordinal utility theory which assumes that timeless replacement of perfect substitutes leads to no change in any individual's welfare as in Proposition 1. In Proposition 1, l_d at each date $d = 0,1,2,3,$ … represents a loan to a market participant e.g. a firm. However, instead of representing a loan to the firm, it can be a shareholder's stake in the firm returned to him/her with its investment return at each date $d = 0,1,2,3,$ … respectively, whilst all of his/her returned stake is reinvested in the firm again at each one of these dates. Therefore, Proposition 1 holds, irrespective of whether one assumes l_d is an investment in the form of a loan or an equity stake in a firm, in each case the firm ends up with a free lunch within a finite period. This reinterpretation of Proposition 1 explains the less well-known free lunch for shareholders in the Arrow-Debreu general equilibrium model (Geanakoplos, 1987, p. 119) where: "owners...of firms collect profits even though they have contributed nothing".

Arrow and Debreu (1954) uphold the law of one price in its standard sense, however, Geanakoplos (1987, p. 119) highlights that under perfect foresight (i.e. certainty): "in an Arrow-Debreu equilibrium….there is no…uniform rate of profit". This is a self-contradictory stance in the Arrow-Debreu model, as it implicitly assumes that in the competitive, efficient and frictionless

economy of the Arrow-Debreu model, arbitrage will eliminate any dispersion in the prices of each commodity, but not the dispersion in the rates of profits of different firms producing the same commodity. Kemp (2008, p. 200-201) argues that for the Arrow-Debreu model (1954) to be internally consistent, households must be either “incompletely informed” or “incompletely rational”, neither of which is an assumption that Arrow and Debreu (1954) make. Ackerman and Nadal (2004) present wider critiques of the foundations of general equilibrium theory by different scholars specializing in this area.