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OPTIMALITY OF BUY-AND-HOLD STRATEGIES

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Abstract

The buy-and-hold way of investing has been taken as gospel by many professional investors since the 1960s. In recent years, however, it has come under harsh attack from both academics and practitioners who claim its ineffectiveness in the face of increasingly volatile markets. This research takes a theoretical approach to evaluating its effectiveness by invoking a powerful optimality theorem to gauge its effectiveness or, more specifically, its optimality level. In terms of optimality level, we determine how well it fares against three other popular strategies – lock-in, random-timing, and stop-loss. To make the concept of optimality level practically operational, we set up a two-factor model to depict the market environment and use Monte Carlo simulation to determine the optimality levels of these strategies. In terms of average optimality level, our results show that, in general, buy-and-hold strategies outperform the other three strategies in stable market environment, but they are outperformed by lock-in and stop-loss strategies in volatile market environment.

Keywords: Buy-and-Hold, Optimality Level, Strategies, Two-Factor Model, Simulation

1. Introduction

A buy-and-hold strategy, simply stated, is a passive investment technique by which an investor buys stocks and holds them for a long time, regardless of market conditions. The term “buy and hold” appears to have been coined by Alexander (1961), who came up with the term in his analysis of stock price data. According to Fama (1965) and Black (1971), if a financial market is efficient and thus every security is fairly priced, then it is to the advantage of investors not to trade frequently and, instead, to buy a well-diversified portfolio and hold on to it. In fact, the initial use of buy-and-hold can be traced back to the landmark research of Cowles (1933), who tested the Dow Theory against a buy-and-hold strategy of buying and holding a well-diversified portfolio. His findings show that the Dow Theory would have yielded an annual return of 12 percent while the buy-and-hold strategy an annual return of 15.5 percent.

Since Cowles' (1933) research, many studies[1] (Alexander, 1964; Fama, 1965; Fama and Blume, 1966; Jensen and Benington, 1970; Black, 1971; Sweeney, 1988; Siegel, 2002) have followed suit and used buy-and-hold strategy as a benchmark to investigate the behavior of stock prices, the effectiveness of investment strategies, the efficiency of stock markets, and so on. In the investment world, the buy-and-hold way of investing has been considered as gospel by many professional investors for nearly six decades. In his best-selling finance book, *A Random Walk Down Wall Street*, Burton Malkiel (2003, p. 15), an emeritus economics professor at Princeton University, hails outright the merit of buy-and-hold: "Investors would be far better off buying and holding an index fund than attempting to buy and sell individual securities or actively managed mutual funds. I boldly stated that buying and holding all the stocks in a broad stock-market average – as index funds do – was likely to outperform professionally managed funds whose high expense charges and large trading costs detract substantially from investment returns."

In recent years, buy-and-hold has been subjected to serious attack from both academics and practitioners (Becker and Seshadri, 2003; Bansal *et al.* 2004; Lo, 2012; Wallace, 2012; Zamansky, 2012; Minnucci, 2015), who claim the ineffectiveness of the strategy in the face of increasingly volatile markets. According to Andrew Lo (2012), a noted finance professor at MIT, buy-and-hold works only in stable financial environment, such as those from the 1940s to the early 2000s. However, the financial environment over the last 20 years has become so volatile that it renders the strategy ineffective. Lo (see Wallace, 2012) proclaims that "Buy-and-hold doesn't work anymore. The volatility is too significant. Almost any asset can suddenly become much more risky. Buying into a mutual fund and holding it for 10 years is no longer going to deliver the same kind of expected return that we saw over the course of the last seven decades, simply because of the nature of financial markets and how complex it's gotten."

In terms of investment effectiveness, is buy-and-hold good enough to be qualified as a benchmark against which all other investment strategies are evaluated? In fact, rarely has buy-and-hold itself been subjected to theoretical scrutiny to determine its qualification as a benchmark for other strategies. This research is intended to do just that. In this regard, we invoke a powerful optimality theorem (see Hirshleifer, 1970; Ingersoll, 1987; Dybvig, 1988) to investigate the effectiveness or, more specifically, the optimality of buy-and-hold. Simply put, this optimality theorem says that, in a complete and perfect market environment, an investment strategy is optimal if its terminal values are in reverse order of the terminal state price densities, where state price density is defined as state price divided by state probability. That said, given an arbitrary non-optimal strategy (say A) with a cost of Y_0^A at time 0, we rearrange the terminal values of A in reverse order of the terminal state price densities to obtain the corresponding optimal strategy (say O). We call the cost (say Y_0^O) of strategy O at time 0 the amount of optimality or the optimality level of A. We will show in the next section that Y_0^O is less than or at most equal to Y_0^A. In other words, given that strategies A and O both produce exactly the same distribution of terminal values, investors pay less with strategy O than with strategy A. In this study, we set the cost of strategy A at $1 at time 0 and thus calibrate its optimality level to have a value from 0 to 1, where 0 stands for nil optimality and 1 absolute optimality. That is, strategies with larger optimality level have greater amount of optimality. We use the following example to illustrate the meaning of the optimality level.

Consider a one-period strategy A. Let there be three equally probable states 1, 2, and 3 at time 1 with state 1 price = 0.1, state 2 price = 0.15, and state 3 price = 0.2. Then, the state price density is 0.3 for state 1, 0.45 for state 2, and 0.6 for state 3. At time 1, suppose strategy A is that its value is 1 if state 1 occurs, 2 if state 2 occurs, and 3 if state 3 occurs. Then the cost of A at time 0 is (0.1 x 1) + (0.15 x 2) + (0.2 x 3) = 1.0. If we rearrange the three values of A at time 1 in reverse order of the three state price densities, then we obtain the corresponding optimal strategy O whose value is 3 if state 1 occurs, 2 if state 2 occurs, and 1 if state 3 occurs. The cost of O at time 0 is (0.1 x 3) + (0.15 x 2) + (0.2 x 1) = 0.8. That is, the optimality level of A is 0.8, which means strategy A is only 80% optimal. Table 1 gives a visual presentation of strategy A and the corresponding optimal strategy O.

[1] See Bernstein (1992) for a thorough discussion of the contributions of these studies to financial markets.

Table 1. Strategy A and the corresponding optimal strategy O

State	Probability	State price	State price density	Value of A	Value of O
1	$\frac{1}{3}$	0.10	0.30	1	3
2	$\frac{1}{3}$	0.15	0.45	2	2
3	$\frac{1}{3}$	0.20	0.60	3	1

Given that buy-and-hold involves only one buy trade at time 0 and one sell trade at time T, one major advantage of the strategy is that it minimizes transaction costs. In this study, we test three other popular strategies (i.e., lock-in, random-timing, and stop-loss), each of which involves only a small number of trades, to determine how each of them measures up to buy-and-hold in terms of optimality level. To make the concept of optimality level practically operational, we set up a two-factor model to depict the dynamic and stochastic nature of the market environment – one factor for the price of a portfolio of risky securities and one factor for the spot interest rate. We employ Monte Carlo simulation to generate 100,000 simulation runs for each strategy and then determine the optimality levels of these strategies.

The rest of the paper is organized as follows: Section 2 gives the theoretical basis for the optimality level. In Section 3, we formulate a two-factor model to depict the market environment and develop a multi-period formula for the terminal state price density for use in computing optimality level. Section 4 describes the four strategies. Section 5 outlines our simulation design and presents our simulation results. Section 6 concludes this research.

2. Theoretical basis for optimality level

An investment strategy is a financial asset whose values over time can be expressed simply as a value vector. In symbols, we can express the values of a strategy in a multi-period setting as $\tilde{Y} = \left[\tilde{Y_1}, \tilde{Y_2}, \ldots, \tilde{Y_T}\right]$, where each $\tilde{Y_t}$ (t = 1, 2, …, T) is random because its value at time t depends on which of the states occurs at time t. In a complete and perfect market with m mutually exclusive and exhaustive states at terminal time T, a strategy can be expressed as a linear combination of the m state securities and thus the cost Y_0 of a strategy at time 0 can be written as

$$Y_0 = \sum_{i=1}^{m} s_{0T}(i) Y_T(i) \tag{1}$$

where $s_{0T}(i)$ is the state price for state i and $Y_T(i)$ is the terminal value of the strategy at time T in state i, where i = 1, 2, …, m. Let $p_{0T}(i)$ be the probability of state i and $d_{0T}(i) = \frac{s_{0T}(i)}{p_{0T}(i)}$ be the state price density for state i. Then the cost of a strategy in (1) can also be written as

$$Y_0 = \sum_{i=1}^{m} \left(\frac{s_{0T}(i)}{p_{0T}(i)} Y_T(i)\right) p_{0T}(i) = \sum_{i=1}^{m} (d_{0T}(i) Y_T(i))\, p_{0T}(i) = E_0^P \left[\tilde{d}_{0T} \tilde{Y_T}\right] \tag{2}$$

Suppose an investor begins with a wealth Y_0 at time 0 and has a utility function $U(\cdot)$, which is increasing and strictly concave. He wants to maximize a von Neumann-Morgenstern utility function subject to his wealth constraint. In this situation, to maximize $\sum_{i=1}^{m} p_{0T}(i) U(Y_T(i))$ subject to $\sum_{i=1}^{m} s_{0T}(i) Y_T(i) = Y_0$, we use the Lagrange multiplier method as follows:

$$L = \sum_{i=1}^{m} p_{0T}(i) U(Y_T(i)) - \lambda \left[\sum_{i=1}^{m} s_{0T}(i) Y_T(i) - Y_0\right] \tag{3}$$

where λ is the Lagrange multiplier. To obtain his optimal choice of Y_T's, we take the partial derivatives with respect to $Y_T(1)$, $Y_T(2)$, …, $Y_T(m)$.

$$\frac{\partial L}{\partial Y_T(i)} = p_{0T}(i) \frac{\partial U(Y_T(i))}{\partial Y_T(i)} - \lambda s_{0T}(i) = 0 \tag{4}$$

We have $\frac{\partial U(Y_T(i))}{\partial Y_T(i)} = \lambda \frac{s_{0T}(i)}{p_{0T}(i)} = \lambda d_{0T}(i)$, where $d_{0T}(i)$ is the state price density for state *i* at terminal time T. Since $d_{0T} > 0$, $\frac{\partial U}{\partial Y_T} > 0$, and $\frac{\partial^2 U}{\partial Y_T^2} < 0$, the terminal value Y_T is non-increasing in the terminal state price density d_{0T}. That is, a strategy is optimal only if $d_{0T}(k) > d_{0T}(l)$ implies $Y_T(k)$ < $Y_T(l)$, for any two terminal states *k* and *l*, where $1 \leq k < l \leq m$.

Using this inverse relation between terminal value and terminal state price density, we can quantify the amount of optimality of a strategy. Suppose an arbitrary non-optimal strategy A has *m* pairs of observations for its terminal value and state price density: $[Y_T^A(1), d_{0T}(1)]$, $[Y_T^A(2), d_{0T}(2)]$, …, $[Y_T^A(m), d_{0T}(m)]$. If we rearrange Y_T^A's in reverse order of d_{0T}'s to obtain the corresponding strategy (say O), then strategy O is optimal because $d_{0T}(k) > d_{0T}(l)$ implies $Y_T^O(k)$ < $Y_T^O(l)$, for any two terminal states *k* and *l*, where $1 \leq k < l \leq m$. Hence, given *m* equally probable terminal states, the costs of strategies A and O at time 0 are respectively $Y_0^A = \frac{1}{m}\sum_{i=1}^{m} d_{0T}(i) Y_T^A(i)$ and $Y_0^O = \frac{1}{m}\sum_{i=1}^{m} d_{0T}(i) Y_T^O(i)$. In other words, Y_0^O is the optimality level of strategy A.

Now we show that the cost of strategy A is larger than that of the corresponding optimal strategy O. That is, Y_0^A **>** Y_0^O. Suppose strategy A has only two (say *k* and *l*) of the *m* equally probable states where $d_{0T}(k) > d_{0T}(l)$ but $Y_T(k) > Y_T(l)$. If we switch $Y_T(k)$ and $Y_T(l)$ between terminal states *k* and *l*, then we have

$$\begin{aligned} Y_0^A - Y_0^O &= [s_{0T}(k)Y_T(k) + s_{0T}(l)Y_T(l)] - [s_{0T}(k)Y_T(l) + s_{0T}(l)Y_T(k)] \\ &= [Y_T(k) - Y_T(l)][s_{0T}(k) - s_{0T}(l)] \\ &= \frac{1}{m}[Y_T(k) - Y_T(l)][d_{0T}(k) - d_{0T}(l)] > 0 \end{aligned} \quad (5)$$

where $s_{0T}(i) = d_{0T}(i) p_{0T}(i) = \frac{d_{0T}(i)}{m}$ and *i* = *k* or *l*.

3. Model setup

In this study, we assume that the market environment is governed by a two-factor model – one factor for the price of a portfolio of risky securities (e.g., a stock market index) and one factor for the spot interest rate. We use a lognormal diffusion process[2] (see Samuelson, 1965; Black and Scholes, 1973) to depict the price $S(t)$ of the portfolio and an Ornstein-Uhlenbeck (O-U) process (see Uhlenbeck and Ornstein, 1930; Bhattacharya and Waymire, 2009; Kung and Wu, 2013) to depict the spot interest rate $r(t)$.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \quad (6)$$

$$dr(t) = \theta(\mu - r(t))dt + \sigma_r dW_r(t) \quad (7)$$

where α is the expected return on the portfolio, σ is the volatility of the portfolio, σ_r is the volatility of the spot interest rate, $W(t)$ and $W_r(t)$ are standard Wiener processes, and the correlation between the two Wiener processes is ρ. In (7), the spot rate has a tendency to revert to its long-term mean value μ at a rate of θ. This mean-reverting phenomenon is evident for interest rates. When interest rates are high, the demand for funds will decrease and eventually they will fall. When interest rates are low, the demand for funds will increase and eventually they will rise.

With (6) and (7), we now derive the terminal state price density for use in computing the optimality level. We convert the random term on the right-hand side of (6) to one without $S(t)$. Setting $X(t) = ln\, S(t)$ and applying Ito's theorem, the diffusion process for $X(t)$ is

[2] The model in (6), better known as the geometric Brownian motion, to depict the price $S(t)$ was developed by Samuelson (1965).

$$dX(t) = \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma dW(t) \tag{8}$$

The corresponding risk-neutral diffusion processes for $X(t)$ and $r(t)$ in (8) and (7) are

$$dX(t) = \left(r(t) - \frac{\sigma^2}{2}\right)dt + \sigma dW(t) \tag{9}$$

$$dr(t) = \theta\left(\mu - \frac{\gamma\sigma_r}{\theta} - r(t)\right)dt + \sigma_r dW_r(t) \tag{10}$$

where γ is the market price of interest rate risk. The diffusion processes for $X(t)$ and $r(t)$ in (8) and (7) can be written in discrete form as follows:

$$X(t+\Delta t) = X(t) + \left(\alpha - \frac{\sigma^2}{2}\right)\Delta t + \sigma\omega\sqrt{\Delta t} \tag{11}$$

$$r(t+\Delta t) = r(t) + \theta(\mu - r(t))\Delta t + \sigma_r\omega_r\sqrt{\Delta t} \tag{12}$$

where $\Delta t = t_j - t_{j-1}$ (j = 1, 2, …, n), and ω and ω_r are two correlated standard normal variables. These two variables are obtained by drawing two independent values ε_1 and ε_2 from a standard normal distribution and letting $\omega = \varepsilon_1$ and $\omega_r = \varepsilon_1\rho + \varepsilon_2\sqrt{1-\rho^2}$, where ρ is the correlation between ω and ω_r.

Similarly, the risk-neutral diffusion processes for $X(t)$ and $r(t)$ in (9) and (10) can be written in discrete form as follows:

$$X(t+\Delta t) = X(t) + \left(r(t) - \frac{\sigma^2}{2}\right)\Delta t + \sigma\omega\sqrt{\Delta t} \tag{13}$$

$$r(t+\Delta t) = r(t) + \theta\left(\mu - \frac{\gamma\sigma_r}{\theta} - r(t)\right)\Delta t + \sigma_r\omega_r\sqrt{\Delta t} \tag{14}$$

Given $X(t)$ and $r(t)$ in (11) and (12), we set $Z_j = \begin{bmatrix} X(t_j) \\ r(t_j) \end{bmatrix}$ and $\alpha^P = \begin{bmatrix} \alpha_X^P \\ \alpha_r^P \end{bmatrix} = \begin{bmatrix} E^P(X(t_j)) \\ E^P(r(t_j)) \end{bmatrix} = \begin{bmatrix} X(t_{j-1}) + \left(\alpha - \frac{\sigma^2}{2}\right)\Delta t \\ r(t_{j-1}) + \theta(\mu - r(t_{j-1}))\Delta t \end{bmatrix}$. The variance-covariance (V-C) matrix and its inverse V-C matrix are

$$\Sigma = \Delta t \begin{bmatrix} \sigma^2 & \rho\sigma\sigma_r \\ \rho\sigma\sigma_r & \sigma_r^2 \end{bmatrix} \text{ and } \Sigma^{-1} = \frac{1}{\Delta t\sigma^2\sigma_r^2(1-\rho^2)} \begin{bmatrix} \sigma_r^2 & -\rho\sigma\sigma_r \\ -\rho\sigma\sigma_r & \sigma^2 \end{bmatrix} \tag{15}$$

That is, under the objective probability P, Z_j is distributed as $N_2(\alpha^P, \Sigma)$. Hence, the objective transition probability density $p_{(j-1)j} = p_{(j-1)j}(Z_j)$ over the time period $\Delta t = t_j - t_{j-1}$ (where j = 1, 2, …, n) is

$$p_{(j-1)j} = \frac{1}{\sqrt{(2\pi)^2|\Sigma|}} exp\left[-\frac{(Z_j - \alpha^P)^T\Sigma^{-1}(Z_j - \alpha^P)}{2}\right] \tag{16}$$

where $(Z_j - \alpha^P)^T\Sigma^{-1}(Z_j - \alpha^P)$ equals

$$\frac{1}{1-\rho^2}\left[\left(\frac{X(t_j)-\alpha_X^P}{\sigma}\right)^2 + \left(\frac{r(t_j)-\alpha_r^P}{\sigma_r}\right)^2 - 2\rho\left(\frac{X(t_j)-\alpha_X^P}{\sigma}\right)\left(\frac{r(t_j)-\alpha_r^P}{\sigma_r}\right)\right] \tag{17}$$

From the stationary independent increment assumption of Brownian motion, the joint transition probability density function under the objective probability *P* is

$$\prod_{j=1}^{n} p_{(j-1)j} = \frac{1}{(2\pi)^n\sqrt{|\Sigma|^n}} exp\left[-\sum_{j=1}^{n}\frac{\left(Z_j-\alpha^P\right)^T\Sigma^{-1}\left(Z_j-\alpha^P\right)}{2}\right] \quad (18)$$

Given $X(t)$ and $r(t)$ in (13) and (14), we set $\alpha^Q = \begin{bmatrix}\alpha_X^Q\\ \alpha_r^Q\end{bmatrix} = \begin{bmatrix}E^Q\left(X(t_j)\right)\\ E^Q\left(r(t_j)\right)\end{bmatrix} = \begin{bmatrix}X(t_{j-1}) + \left(r(t_{j-1}) - \frac{\sigma^2}{2})\right)\Delta t\\ r(t_{j-1}) + \theta\left(\mu - \frac{\gamma\sigma_r}{\theta} - r(t_{j-1})\right)\Delta t\end{bmatrix}$. That is, under the risk-neutral probability Q, Z_j is distributed as $N_2(\alpha^Q,\Sigma)$. Hence, the risk-neutral transition probability density $q_{(j-1)j} = q_{(j-1)j}\left(Z_j\right)$ over the time interval $\Delta t = t_j - t_{j-1}$ (where *j* = 1, 2, …, *n*) is

$$q_{(j-1)j} = \frac{1}{\sqrt{(2\pi)^2|\Sigma|}} exp\left[-\frac{(Z_j-\alpha^Q)^T\Sigma^{-1}(Z_j-\alpha^Q)}{2}\right] \quad (19)$$

where $(Z_j - \alpha^Q)^T\Sigma^{-1}(Z_j - \alpha^Q)$ equals

$$\frac{1}{1-\rho^2}\left[\left(\frac{X(t_j)-\alpha_X^Q}{\sigma}\right)^2 + \left(\frac{r(t_j)-\alpha_r^Q}{\sigma_r}\right)^2 - 2\rho\left(\frac{X(t_j)-\alpha_X^Q}{\sigma}\right)\left(\frac{r(t_j)-\alpha_r^Q}{\sigma_r}\right)\right] \quad (20)$$

The joint transition probability density function under the risk-neutral probability *Q* is

$$\prod_{j=1}^{n} q_{(j-1)j} = \frac{1}{(2\pi)^n\sqrt{|\Sigma|^n}} exp\left[-\sum_{j=1}^{n}\frac{\left(Z_j-\alpha^Q\right)^T\Sigma^{-1}\left(Z_j-\alpha^Q\right)}{2}\right] \quad (21)$$

Let *P* and *Q* be equivalent probability measures. For every ordered time steps $0 \equiv t_0 < t_1 <\ldots< t_n \equiv T$, the Radon-Nikodym derivative $\frac{dQ}{dP}$ up to terminal time *T* is defined to be the limit of the likelihood ratio

$$\frac{dQ}{dP} = \lim_{n\to\infty}\frac{\prod_{j=1}^{n} q_{(j-1)j}}{\prod_{j=1}^{n} p_{(j-1)j}} \quad (22)$$

For small time period $\Delta t = t_j - t_{j-1}$ (*j* = 1, 2,…, *n*), the Radon-Nikodym derivative $\frac{dQ}{dP}$ up to terminal time *T* can be written in discrete form as

$$\prod_{j=1}^{n}\left(\frac{q_{(j-1)j}}{p_{(j-1)j}}\right) = \exp\left[\frac{1}{2}\sum_{j=1}^{n}\left\{\left(Z_j - \alpha^P\right)^T\Sigma^{-1}\left(Z_j - \alpha^P\right) - \left(Z_j - \alpha^Q\right)^T\Sigma^{-1}\left(Z_j - \alpha^Q\right)\right\}\right] \quad (23)$$

Hence, the terminal state price density for a strategy under the O-U process from time $0 \equiv t_0$ to time $T \equiv t_n$ is

$$\Phi_{0T} = \prod_{j=1}^{n}\Phi_{(j-1)j} = \prod_{j=1}^{n}\frac{\left(\frac{q_{(j-1)j}}{p_{(j-1)j}}\right)}{1+r(t_{j-1})\Delta t} \quad (24)$$

4. The four strategies

Besides buy-and-hold strategies, we also investigate the optimality of lock-in, random-timing, and stop-loss strategies. We assume the investor starts out with \$1 (\$1 can stand for a million or a billion dollars) for investment at initial time 0. For each of the four strategies, the investor will allocate his funds between a risky portfolio (which can be an index fund) and a zero-coupon bond. The price $B[\cdot]$ of a zero-coupon bond at time t_j paying \$1 at terminal time T under the Ornstein-Uhlenbeck process is

$$B\left[r(t_j), t_j, T\right] = exp\left\{D\left[C - r(t_j)\right] - C\left[T - t_j\right] - \frac{\sigma_r^2 D^2}{4\theta}\right\} \tag{25}$$

where $C = \mu + \frac{\gamma\sigma_r}{\theta} - \frac{\sigma_r^2}{2\theta^2}$ and $D = \frac{1-exp\left[-\theta(T-t_j)\right]}{\theta}$.

With a buy-and-hold strategy, the investor invests a fraction (say *a*) of his initial amount of \$1 in the risky portfolio at time 0 and the remaining \$1 – *a* in a zero-coupon bond to mature at time T. He will hold the portfolio until time T, at which time his wealth is the sum of his holding of the portfolio and that of the zero-coupon bond. In respect of the portfolio, the strategy involves a buy trade at time 0 and a sell trade at time T. For implementation, the fraction *a* is set at 0.8, 0.9, and 1.0. For example, if *a* is set at 0.8, he will invest at time 0 \$0.8 in the portfolio and \$0.2 in a zero-coupon bond to mature at time T.

With a lock-in strategy, the investor invests his initial amount of \$1 in the risky portfolio at time 0 and holds it until its value goes up to a limit value (say *b*) or above, at which point he switches from the portfolio to a zero-coupon bond to mature at time T. In respect of the risky portfolio, the strategy involves a buy trade at time 0 and a sell trade at some time between time 0 and time T. For implementation, the limit value *b* is set at 1.10, 1.20, and 1.30 of the initial amount of \$1. For example, if *b* is set at 1.20, he will switch from the portfolio to the zero-coupon bond when its value goes up to \$1.20 or above. As long as its value is less than \$1.20, he will continue to hold the portfolio.

With a random-timing strategy, the investor will hold the portfolio for a fraction (say *c*) of his investment horizon [0,T] and a zero-coupon bond for the remaining fraction 1 – *c* of [0,T]. For simplicity, we divide [0,T] into 10 equal intervals. Accordingly, he will hold the portfolio in any *e* intervals such that $c = \frac{e}{10}$ (where 0 < *e* < 10) and a zero-coupon bond in the remaining 10 - *e* intervals such that $1 - c = \frac{10-e}{10}$. The choice between the portfolio and the zero-coupon bond at the beginning of each interval is based on the outcome of a toss of an unbiased coin. For implementation, the fraction *c* is set at 0.4, 0.6, and 0.8. Suppose T = 5 years and *c* = 0.4, then the five-year horizon is divided into ten 6-month intervals and the investor will hold the portfolio in four (0.4 x 10) randomly selected intervals and the bond in the remaining six intervals. For example, a possible random ordering of four intervals for the portfolio and six intervals for the bond is P-B-B-B-P-P-B-B-P-B, where P is the interval for the portfolio and B is the interval for the bond. Given that, in respect of the portfolio, we have a buy trade at the start of interval 1, then a sell trade at the end of interval 1, then a buy trade at the start of interval 5, then a sell trade at the end of interval 6, then a buy trade at the start of interval 9, and finally a sell trade at the end of interval 9. The strategy involves a total of six buy-sell trades as it evolves through the investment horizon.

With a stop-loss strategy, the investor invests \$1 in the risky portfolio at time 0 and holds it until its value drops to a limit value (say *d*) or below, at which point he switches from the portfolio to a zero-coupon bond to mature at time T. Hence, the strategy involves a buy trade at time 0 and a sell trade at some time between time 0 and time T. For implementation, the limit value *d* is set at 0.7, 0.8, and 0.9 of the initial amount of \$1. For example, if *d* is set at 0.8, he will switch from the portfolio to a zero-coupon bond when its value drops to \$0.8 or below. As long as its value is greater than \$0.8, the investor will continue to hold the portfolio.

5. Simulation design and results

5.1. Simulation design

To determine the optimality levels of the four strategies, we use a value of 0.2 for the σ in (6) to depict a stable market environment[3] and a value of 0.4 to depict a volatile market environment, supplemented with either 0.01 or 0.02 for the σ_r in (7). These values for σ and σ_r are reasonable in practical sense. In respect of σ, the volatility of the S&P 500 Index, for example, is close to 0.2 over the ten years from 2006 to 2015, and close to 0.4 over the turbulent period from September 2008 to December 2008. In respect of σ_r, the volatility of U.S. short-term interest rates, for example, fluctuates mostly between 0.01 and 0.02. Accordingly, we use the following values (see Hull, 2015) for the five parameters in (6) and (7): α = 0.08 and 0.10, σ = 0.2 and 0.4, σ_r = 0.01 and 0.02, θ = 0.1, μ = 0.06, and ρ = -0.2. Altogether, we have eight (2 values for α, 2 for σ, and 2 for σ_r) different scenarios for the market environment, which should be large enough to encompass a wide range of possible market conditions. We will determine the optimality levels of the four strategies with the investment horizon being set at 0.5 year, 1 year, 3 years, 5 years, and 10 years.

To ensure a high degree of precision, we simulate 100,000 independent paths for each strategy to obtain 100,000 pairs of observations for terminal value and terminal state price density: $[Y_T(1), d_{0T}(1)]$, $[Y_T(2), d_{0T}(2)]$, …, $[Y_T(100000), d_{0T}(100000)]$. With the 100,000 pairs of observations, we rearrange the terminal values in reverse order of the terminal state price densities to obtain the corresponding optimal strategy O where $d_{0T}(k) > d_{0T}(l)$ implies $Y_T^O(k) < Y_T^O(l)$ for any two states *k* and *l*, where 1 ≦ *k*, *l* ≦ 100,000 terminal states. Hence, the optimality level of a strategy at time 0 is

$$V_0^O = \frac{1}{100000}\sum_{i=1}^{100000} d_{0T}(i) Y_T^O(i) \qquad (26)$$

5.2. Simulation results

Tables 2 through 5 report the optimality levels of the four strategies under eight different market conditions. The upper part of each table is for α = 0.08 and the lower part of each table is for α = 0.10. Note that the optimality level of a strategy has a value between 0 and 1 such that strategies with larger optimality levels are more effective and thus ranked higher than those with smaller optimality levels.

Regardless of market conditions, the size of the investment horizon has an adverse impact on their optimality levels. Specifically, their optimality levels become smaller as investment horizon T is longer. In addition, the value of α, the expected return on the risky portfolio, in (6) has a significant effect on the optimality levels of buy-and-hold strategies. In general, the larger the value of α, the larger their optimality levels. Given that this study is about the optimality of buy-and-hold strategies, we focus on how well they fare against the other three strategies based on their average optimality levels.

Table 2 shows the optimality levels of the four strategies with σ = 0.2 and σ_r = 0.01, which depicts a stable securities market with mild interest rate. In such a market environment, buy-and-hold strategies, in general, excel the other three strategies in terms of average optimality level (except for the case where T = 0.5 year and α = 0.08), particularly when α = 0.10. Between lock-in and stop-loss, the former performs slightly better than the latter when α = 0.08, but the latter performs obviously better than the former when α = 0.10. Random-timing strategies are ranked at the bottom. For example, when T = 5 years and α = 0.10, the average optimality level is 0.9781 for buy-and-hold, 0.9664 for stop-loss, 0.9510 for lock-in, and 0.9469 for random-timing.

[3] A stable market environment means a market with reasonable degree of volatility.

Table 2. Optimality levels in a stable securities market with mild interest rate

	Strategy	Limit	0.5-Year	1-Year	3-Year	5-Year	10-Year
α = 0.08	Buy-and-Hold	0.80	0.9961	0.9935	0.9794	0.9658	0.9283
		0.90	0.9956	0.9927	0.9765	0.9608	0.9180
		1.00	0.9961	0.9919	0.9739	0.9567	0.9132
		Average	0.9959	0.9927	0.9766	0.9611	0.9198
	Lock-In	1.10	0.9964	0.9922	0.9760	0.9601	0.9201
		1.20	0.9965	0.9925	0.9740	0.9570	0.9124
		1.30	0.9964	0.9925	0.9741	0.9568	0.9116
		Average	0.9964	0.9924	0.9747	0.9580	0.9147
	Random-Timing	0.40	0.9946	0.9896	0.9693	0.9479	0.9005
		0.60	0.9947	0.9895	0.9687	0.9471	0.8974
		0.80	0.9951	0.9899	0.9703	0.9495	0.8999
		Average	0.9948	0.9897	0.9694	0.9482	0.8992
	Stop-Loss	0.70	0.9953	0.9911	0.9753	0.9597	0.9119
		0.80	0.9955	0.9915	0.9758	0.9599	0.9121
		0.90	0.9961	0.9923	0.975	0.9605	0.9132
		Average	0.9956	0.9916	0.9757	0.9600	0.9124
α = 0.10	Buy-and-Hold	0.80	0.9977	0.9959	0.9885	0.9810	0.9607
		0.90	0.9974	0.9953	0.9868	0.9780	0.9546
		1.00	0.9978	0.9952	0.9854	0.9755	0.9507
		Average	0.9977	0.9955	0.9869	0.9781	0.9553
	Lock-In	1.10	0.9970	0.9923	0.9720	0.9488	0.8936
		1.20	0.9978	0.9943	0.9735	0.9516	0.8932
		1.30	0.9978	0.9950	0.9746	0.9526	0.8944
		Average	0.9975	0.9939	0.9734	0.9510	0.8937
	Random-Timing	0.40	0.9935	0.9874	0.9629	0.9368	0.8804
		0.60	0.9945	0.9891	0.9673	0.9453	0.8970
		0.80	0.9959	0.9916	0.9751	0.9586	0.9189
		Average	0.9946	0.9894	0.9684	0.9469	0.8988
	Stop-Loss	0.70	0.9975	0.9951	0.9847	0.9724	0.9322
		0.80	0.9975	0.9952	0.9833	0.9694	0.9243
		0.90	0.9976	0.9941	0.9763	0.9572	0.9020
		Average	0.9975	0.9948	0.9814	0.9664	0.9195

Table 3 shows their optimality levels with σ = 0.4 and σ_r = 0.01, which depicts a volatile securities market accompanied by mild interest rate. In such a volatile market environment, the value of α plays an important role in determining the ranking of buy-and-hold in relation to the other three. Buy-and-hold strategies rank behind both lock-in and stop-loss strategies when α = 0.08, but buy-and-hold strategies rank first, across the board, when α = 0.10. For example, given T = 10 years, the average optimality level, in decreasing order, is 0.9316 for lock-in, 0.9178 for stop-loss, 0.8921 for buy-and-hold, and 0.8739 for random-timing when α = 0.08; whereas the average optimality level is 0.9502 for buy-and-hold, 0.9301 for stop-loss, 0.9187 for lock-in, and 0.8946 for random-timing when α = 0.10.

Table 3. Optimality levels in a volatile securities market with mild interest rate

	Strategy	Limit	0.5-Year	1-Year	3-Year	5-Year	10-Year
α = 0.08	Buy-and-Hold	0.80	0.9960	0.9925	0.9763	0.9584	0.8997
		0.90	0.9955	0.9915	0.9730	0.9525	0.8862
		1.00	0.9960	0.9908	0.9703	0.9490	0.8904
		Average	0.9958	0.9916	0.9732	0.9533	0.8921
	Lock-In	1.10	0.9963	0.9927	0.9804	0.9699	0.9428
		1.20	0.9965	0.9923	0.9760	0.9626	0.9289
		1.30	0.9965	0.9922	0.9748	0.9601	0.9231
		Average	0.9964	0.9924	0.9770	0.9642	0.9316
	Random-Timing	0.40	0.9946	0.9893	0.9671	0.9428	0.8808
		0.60	0.9946	0.9892	0.9663	0.9401	0.8710
		0.80	0.9950	0.9896	0.9672	0.9411	0.8698
		Average	0.9947	0.9894	0.9669	0.9413	0.8739
	Stop-Loss	0.70	0.9959	0.9920	0.9750	0.9591	0.9064
		0.80	0.9961	0.9923	0.9755	0.9612	0.9124
		0.90	0.9965	0.9930	0.9798	0.9706	0.9347
		Average	0.9962	0.9924	0.9767	0.9636	0.9178
α = 0.10	Buy-and-Hold	0.80	0.9977	0.9959	0.9878	0.9795	0.9547
		0.90	0.9974	0.9953	0.9860	0.9763	0.9478
		1.00	0.9978	0.9950	0.9841	0.9742	0.9482
		Average	0.9976	0.9954	0.9860	0.9767	0.9502
	Lock-In	1.10	0.9964	0.9922	0.9770	0.9626	0.9283
		1.20	0.9971	0.9928	0.9740	0.9574	0.9158
		1.30	0.9973	0.9930	0.9736	0.9563	0.9121
		Average	0.9969	0.9928	0.9749	0.9587	0.9187
	Random-Timing	0.40	0.9935	0.9875	0.9626	0.9371	0.8790
		0.60	0.9945	0.9892	0.9672	0.9445	0.8914
		0.80	0.9959	0.9916	0.9746	0.9573	0.9134
		Average	0.9946	0.9894	0.9681	0.9463	0.8946
	Stop-Loss	0.70	0.9977	0.9951	0.9819	0.9697	0.9299
		0.80	0.9977	0.9947	0.9805	0.9683	0.9288
		0.90	0.9973	0.9940	0.9794	0.9682	0.9317
		Average	0.9976	0.9946	0.9806	0.9687	0.9301

Table 4 shows the optimality levels with σ = 0.2 and σ_r = 0.02, which represents a stable securities market agitated by volatile interest rate. In such a market environment, the value of α makes a difference in ranking when we compare buy-and-hold with the other three. As pointed out above, the larger the value of α, the larger the optimality levels of buy-and-hold strategies. When α = 0.08, buy-and-hold strategies rank behind lock-in strategies and, in three out of the five investment horizons, behind stop-loss strategies. When α = 0.10, buy-and-hold strategies rank first in four out of the five investment horizons, trailing only behind lock-in when T = 0.5 year.

Table 4. Optimality levels in a stable securities market with volatile interest rate

	Strategy	Limit	0.5-Year	1-Year	3-Year	5-Year	10-Year
α = 0.08	Buy-and-Hold	0.80	0.9887	0.9791	0.9412	0.9061	0.8148
		0.90	0.9872	0.9763	0.9329	0.8924	0.7894
		1.00	0.9878	0.9730	0.9239	0.8797	0.7716
		Average	0.9879	0.9761	0.9327	0.8928	0.7920
	Lock-In	1.10	0.9905	0.9805	0.9453	0.9081	0.8130
		1.20	0.9896	0.9785	0.9392	0.9011	0.8052
		1.30	0.9892	0.9778	0.9378	0.8990	0.8039
		Average	0.9897	0.9789	0.9408	0.9027	0.8074
	Random-Timing	0.40	0.9887	0.9779	0.9375	0.8947	0.8048
		0.60	0.9877	0.9753	0.9301	0.8845	0.7873
		0.80	0.9871	0.9735	0.9260	0.8772	0.7727
		Average	0.9879	0.9756	0.9312	0.8855	0.7883
	Stop-Loss	0.70	0.9856	0.9730	0.9298	0.8908	0.7860
		0.80	0.9860	0.9742	0.9324	0.8945	0.7911
		0.90	0.9881	0.9780	0.9398	0.9055	0.8036
		Average	0.9866	0.9750	0.9340	0.8969	0.7936
α = 0.10	Buy-and-Hold	0.80	0.9923	0.9861	0.9613	0.9373	0.8730
		0.90	0.9913	0.9842	0.9556	0.9276	0.8551
		1.00	0.9922	0.9827	0.9500	0.9183	0.8417
		Average	0.9920	0.9843	0.9556	0.9278	0.8566
	Lock-In	1.10	0.9925	0.9831	0.9440	0.8990	0.7873
		1.20	0.9929	0.9843	0.9448	0.9024	0.7936
		1.30	0.9927	0.9846	0.9453	0.9025	0.7955
		Average	0.9927	0.9840	0.9447	0.9013	0.7921
	Random-Timing	0.40	0.9893	0.9792	0.9390	0.8952	0.8014
		0.60	0.9895	0.9792	0.9388	0.8981	0.8119
		0.80	0.9903	0.9801	0.9428	0.9047	0.8203
		Average	0.9897	0.9795	0.9402	0.8993	0.8112
	Stop-Loss	0.70	0.9907	0.9824	0.9524	0.9214	0.8298
		0.80	0.9910	0.9833	0.9523	0.9202	0.8233
		0.90	0.9922	0.9842	0.9485	0.9114	0.8037
		Average	0.9913	0.9833	0.9511	0.9177	0.8189

Table 5 shows the optimality levels with σ = 0.4 and σ_r = 0.02, which depicts a volatile securities market aggravated by volatile interest rate. Such a volatile market environment evidently works against buy-and-hold strategies. A change of α from 0.08 to 0.10 does not improve buy-and-hold much in relation to the other three. Of the four strategies, buy-and-hold ranks either third or fourth when α = 0.08 and ranks third when α = 0.10. For example, given T = 3 years, the average optimality level, in decreasing order, is 0.9512 for lock-in, 0.9384 for stop-loss, 0.9223 for random-timing, and 0.9203 for buy-and-hold when α = 0.08; while the average optimality level is 0.9538 for stop-loss, 0.9517 for lock-in, 0.9493 for buy-and-hold, and 0.9369 for random-timing when α = 0.10.

Table 5. Optimality levels in a volatile securities market with volatile interest rate

	Strategy	Limit	0.5-Year	1-Year	3-Year	5-Year	10-Year
α = 0.08	Buy-and-Hold	0.80	0.9882	0.9776	0.9301	0.8811	0.7390
		0.90	0.9867	0.9746	0.9202	0.8638	0.7043
		1.00	0.9875	0.9712	0.9107	0.8507	0.6981
		Average	0.9875	0.9745	0.9203	0.8652	0.7138
	Lock-In	1.10	0.9916	0.9839	0.9588	0.9358	0.8726
		1.20	0.9907	0.9814	0.9489	0.9214	0.8493
		1.30	0.9904	0.9804	0.9457	0.9160	0.8389
		Average	0.9909	0.9819	0.9512	0.9244	0.8536
	Random-Timing	0.40	0.9885	0.9771	0.9316	0.8825	0.7692
		0.60	0.9874	0.9740	0.9212	0.8622	0.7217
		0.80	0.9867	0.9720	0.9141	0.8477	0.6915
		Average	0.9875	0.9744	0.9223	0.8641	0.7275
	Stop-Loss	0.70	0.9869	0.9753	0.9299	0.8918	0.7804
		0.80	0.9877	0.9768	0.9341	0.9001	0.7985
		0.90	0.9902	0.9817	0.9512	0.9306	0.8537
		Average	0.9882	0.9779	0.9384	0.9075	0.8109
α = 0.10	Buy-and-Hold	0.80	0.9921	0.9861	0.9613	0.9373	0.8730
		0.90	0.9910	0.9842	0.9556	0.9276	0.8551
		1.00	0.9921	0.9827	0.9500	0.9183	0.8417
		Average	0.9917	0.9843	0.9556	0.9278	0.8566
	Lock-In	1.10	0.9923	0.9846	0.9570	0.9307	0.8621
		1.20	0.9928	0.9841	0.9497	0.9202	0.8432
		1.30	0.9928	0.9839	0.9483	0.9174	0.8369
		Average	0.9927	0.9842	0.9517	0.9228	0.8474
	Random-Timing	0.40	0.9892	0.9790	0.9376	0.8945	0.8014
		0.60	0.9894	0.9787	0.9356	0.8892	0.8119
		0.80	0.9900	0.9795	0.9374	0.8908	0.8203
		Average	0.9895	0.9790	0.9369	0.8915	0.8112
	Stop-Loss	0.70	0.9907	0.9824	0.9524	0.9222	0.8292
		0.80	0.9910	0.9833	0.9523	0.9252	0.8363
		0.90	0.9922	0.9842	0.9485	0.9349	0.8541
		Average	0.9913	0.9833	0.9511	0.9274	0.8398

6. Conclusion

The buy-and-hold way of investing has been hailed as gospel by many professional investors for nearly six decades. Since the early 2000s, however, it has come under serious siege from both academics and practitioners who claim its ineffectiveness in the face of increasingly volatile markets. This research takes a theoretical approach to evaluating its effectiveness by applying a powerful optimality theorem to examine its effectiveness or, more specifically, its optimality level. In terms of optimality level, we examine how well buy-and-hold fares against three other popular strategies – lock-in, random-timing, and stop-loss. To make the concept of optimality level practically applicable, we set up a two-factor model to depict the market environment and employ Monte Carlo simulation to determine the optimality level of each strategy. In terms of average optimality level, our results show that, in general, buy-and-hold strategies perform better than the other three strategies in stable market environment, but they are outperformed by lock-in and stop-loss strategies in volatile market environment.

In conclusion, one important implication of this research for investment is that, in a volatile market environment, such as what we have seen over the past two decades, the effectiveness of buy-and-hold strategies is clearly in doubt.

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